1. Let X, Y, and Z be random variables with expected values and standard deviations given below:

	Expected Value	Standard Deviation
X	1.5	3.2
Y	0	8.1
Z	6	2.7

Find:

- $E(8 + 2X + Y + Z)$
- $SD(8 + 2X + Y + Z)$

- The expected value of the linear combination is:

$$\begin{aligned} E(8 + 2X + Y + Z) &= 8 + 2E(X) + E(Y) + E(Z) \\ &= 8 + 2 \cdot 1.5 + 0 + 6 \\ &= 17 \end{aligned}$$

- Before computing the standard deviation, note:

$$Var(8 + 2X + Y + Z) = 2^2 Var(X) + Var(Y) + Var(Z)$$

Remember that the standard deviation is the square root of the variance:

$$\begin{aligned} [SD(8 + 2X + Y + Z)]^2 &= 2^2[SD(X)]^2 + [SD(Y)]^2 + [SD(Z)]^2 \\ SD(8 + 2X + Y + Z) &= \sqrt{2^2[SD(X)]^2 + [SD(Y)]^2 + [SD(Z)]^2} \\ &= \sqrt{2^2[3.2]^2 + [8.1]^2 + [2.7]^2} \\ &\approx 10.671 \end{aligned}$$

2. Let X be the the number of crankshafts that fail in a given test of a certain type of vehicle ($X = 0, 1, 2$). Let $Y = 1$ if the clutch fails during that same test and $Y = 0$ otherwise. Consider the joint distribution of X and Y:

$Y \backslash X$	0	1	2
0	0.35	0.1	0.05
1	0.2	0.25	0.05

Find or answer the following:

- $P(X = 1 \text{ and } Y = 1)$
- $P(X = 0)$
- $P(X > 0 \text{ and } Y = 1)$

- The marginal pmfs of X and Y
- Are X and Y independent? Why or why not?

- $P(X = 1 \text{ and } Y = 1) = 0.25$ from the table.
- $P(X = 0)$:

$$\begin{aligned} P(X = 0) &= P(X = 0, Y = 0) + P(X = 0, Y = 1) \\ &= 0.35 + 0.2 \\ &= 0.55 \end{aligned}$$

,

- $P(X > 0 \text{ and } Y = 1)$

$$\begin{aligned} P(X > 0, Y = 1) &= P(X = 1, Y = 1) + P(X = 2, Y = 1) \\ &= 0.25 + 0.05 \\ &= 0.3 \end{aligned}$$

- For the marginal pmf of X, take the row sums of the table:

x	0	1	2
$f_X(x)$	0.55	0.35	0.1

  For the marginal pmf of Y, take the column sums of the table:

y	0	1
$f_Y(y)$	0.5	0.5

- X and Y are independent random variables if and only if $P(X = x, Y = y) = P(X = x) \cdot P(Y = y)$ for all values x and y. That is, the joint pmf must always be the product of the two marginals. However, in this case, $P(X = 1, Y = 1) = 0.25$, while $P(X = 1) \cdot P(Y = 1) = f_X(1) \cdot f_Y(1) = 0.35 \cdot 0.5 = 0.175$. Therefore, X and Y are not independent.

Exercise 2.4. Suppose a standup comedian plans to give a total of $n = 5$ jokes in an entire 2-hour performance. Call a joke a success if at least one audience member laughs. If no audience member laughs, the joke is a failure. Assume that all the jokes are equally funny, with $p = P(\text{success}) = 0.2$. Let X be the random variable that denotes the number of jokes out of the total 5 were successes.

a. Precisely state the distribution of X, giving the values of any parameters necessary.

b. Calculate the probability that the whole night is a failure: i.e., P(no laughs).

c. Calculate the probability that the comedian tells at least 4 successful jokes.

d. Calculated the expected number of successful jokes.

e. Calculate the standard deviation of X.

a. $X \sim \text{Binomial}(n = 5,\ p = 0.2)$

b.

$$\begin{aligned} P(\text{no laughs}) &= P(X = 0) \\ &= \binom{5}{0}(0.2)^0(1 - 0.2)^{5-0} \\ &= (0.8)^5 \\ &= 0.3277 \end{aligned}$$

c.

$$\begin{aligned} P(\text{ at least 4 successful jokes }) &= P(X = 4) + P(X = 5) \\ &= \binom{5}{4}(0.2)^4(1 - 0.2)^{5-4} + \binom{5}{5}(0.2)^5(1 - 0.2)^{5-5} \\ &= 5 \cdot (0.2)^4 \cdot (0.8) + (0.2)^5 \\ &= 0.00672 \end{aligned}$$

d. Expected number of successful jokes $= E(X) = np = 5 \cdot 0.2 = 1$

e. $SD(X) = \sqrt{Var(X)} = \sqrt{np(1 - p)} = \sqrt{5 \cdot 0.2 \cdot (1 - 0.2)} = 0.8944$

Exercise 2.5.
The number of paper jams in a receipt-printer in a grocery store can be modeled as the random variable, $N \sim \text{Poisson}(\lambda = 0.2 \text{ jams per day})$.

a. Find the expected number of jams tomorrow.

b. Find the variance of N.

c. Calculate the probability that there are at most two jams on April 15, 2012.

d. Calculate the probability that there are no jams in the next 7-day week.

e. Let Y be the number of 7-day weeks up to and including the next paper jam. Precisely state the distribution of Y, giving the values of any parameters necessary.

a. Expected number of jams tomorrow $= E(N) = \lambda = 0.2$.

b. $Var(N) = \lambda = 0.2$

c.

$$\begin{aligned}
P(\text{at most 2 jams that day}) &= P(N \leq 2) \\
&= P(N=0) + P(N=1) + P(N=2) \\
&= \frac{e^{-0.2}(0.2)^0}{0!} + \frac{e^{-0.2}(0.2)^1}{1!} + \frac{e^{-0.2}(0.2)^2}{2!} \\
&= 0.8187 + 0.1637 + 0.0164 \\
&= \boxed{0.9988}
\end{aligned}$$

d. First, I need to convert the rate parameter λ into jams per week:

$$\frac{0.2 \text{ jams}}{1 \text{ day}} \times \frac{7 \text{ days}}{1 \text{ week}} = \frac{1.4 \text{ jams}}{1 \text{ week}}$$

Now, I define a new random variable:

$$T \sim \text{Poisson}(\lambda' = 1.4 \text{ jams per week})$$

Here, T is the number of jams next week. Now,

$$\begin{aligned}
P(\text{no jams next week}) &= P(T = 0) \\
&= \frac{e^{-\lambda'}\lambda'^0}{0!} \\
&= e^{-1.4} \\
&= 0.2466
\end{aligned}$$

e. $Y \sim \text{Geometric}(p = P(T \neq 0) = 1 - 0.2466 = 0.7534)$

Exercise 1.1.
Say we have a continuous random variable X with the following pdf:

$$f(x) = \begin{cases} k \cdot x^3 & : 0 \leq x \leq 1 \\ 0 & : x \text{ otherwise.} \end{cases}$$

where k is some real constant.

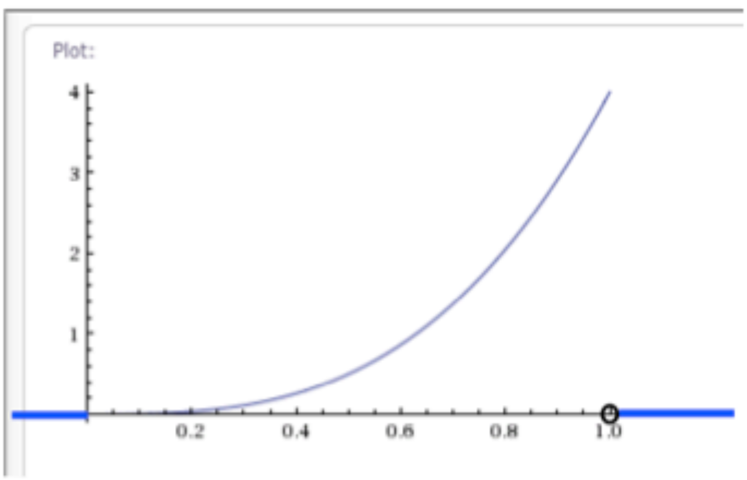
a. Find k such that $f(x)$ is a valid pdf.

We know $\int_{-\infty}^{\infty} f(x)dx = 1$ if the pdf is valid.

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^1 kx^3 dx = \left[\frac{kx^4}{4}\right]_{x=0}^{1} = \frac{k \cdot 1^4}{4} - \frac{k \cdot 0^4}{4} = \frac{k}{4}$$

Hence, $k = 4$.

b. Sketch a graph of $f(x)$ on the Cartesian plane.



c. Find the cdf $F(x)$ of X.

If $x < 0$, then:

$$0 \le F(x) = \int_{-\infty}^{x} f(t)dt \le \int_{-\infty}^{0} f(t)dt = \int_{-\infty}^{0} 0dt = 0$$

And hence, $F(x) = 0$. If $0 \le x \le 1$, then:

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{0} f(t)dt + \int_{0}^{x} f(t)dt = 0 + \int_{0}^{x} 4t^3 dt = x^4$$

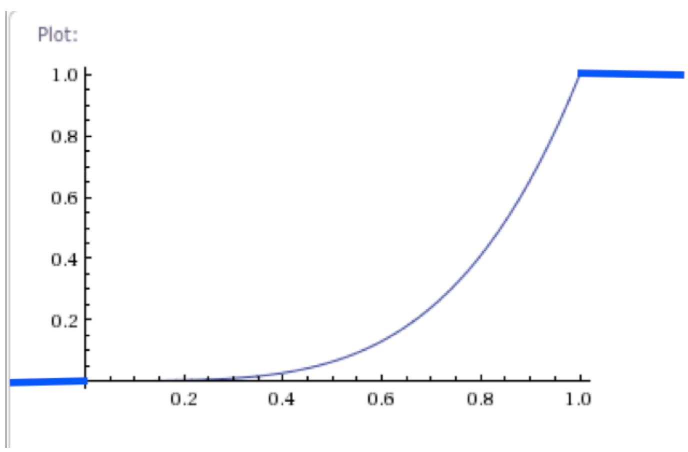
If $x > 1$, then:

$$\begin{aligned} F(x) = \int_{-\infty}^{x} f(t)dt &= \int_{-\infty}^{0} f(t)dt + \int_{0}^{1} f(t)dt + \int_{1}^{x} f(t)dt \\ &= 0 + (x^4)_{x=0}^{1} + 0 \\ &= 1 \end{aligned}$$

Thus,

$$F(x) = \begin{cases} 0 & : x < 0 \\ x^4 & : 0 \le x \le 1 \\ 1 & : x > 1 \end{cases}$$

d. Sketch a graph of $F(x)$ on the Cartesian plane.



e. Find $P(0.2 \leq X \leq 0.8)$

$$P(0.2 \leq X \leq 0.8) = F(0.8) - F(0.2) = 0.8^4 - 0.2^4 = 0.408$$

f. Find $P(X \geq 0.3)$

$$P(X \geq 0.3) = 1 - P(X \leq 0.3) = 1 - F(0.3) = 1 - 0.3^4 = 0.9919$$

g. Find $P(X = 0.5)$

$P(X = 0.5) = P(X \leq 0.5) - P(X < 0.5) = F(0.5) - F(0.5) = 0$

h. Find $E(X)$

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 x \cdot 4 \cdot x^3 = 4\int_0^1 x^4 dx = \frac{4}{5}x^5|_{x=0}^1 = \frac{4}{5}$$

i. Find $Var(X)$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^1 x^2 \cdot 4 \cdot x^3 = 4\int_0^1 x^5 dx = \frac{4}{6}x^6|_{x=0}^1 = \frac{2}{3}$$

Hence:

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{2}{3} - \left(\frac{4}{5}\right)^2 = 0.0267$$

Exercise 1.2.

Say we have a continuous random variable X with the following pdf:

$$f(x) = \begin{cases} k & : 0 \le x \le 5 \\ 0 & : x \text{ otherwise.} \end{cases}$$

where k is some real constant.

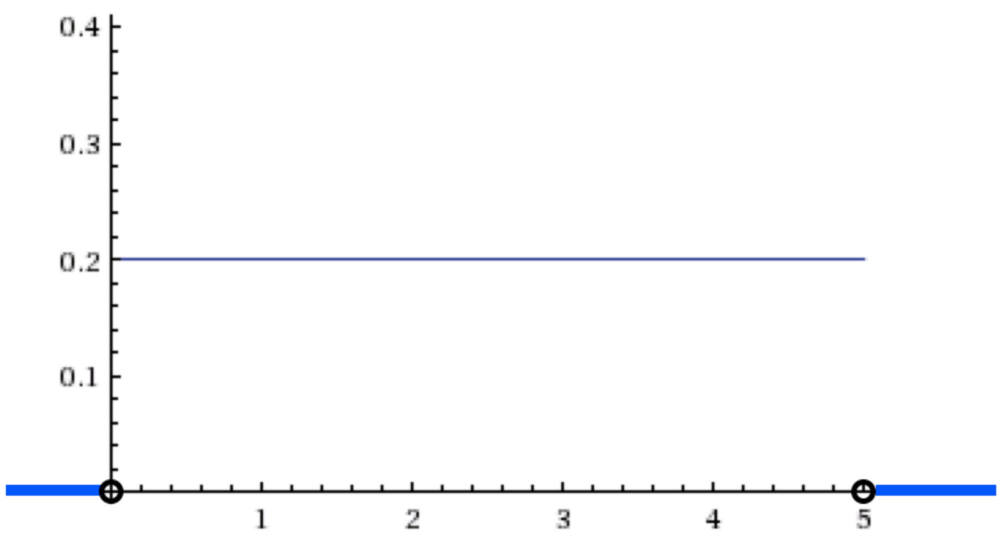
a. Find k such that $f(x)$ is a valid pdf.

We know $\int_{-\infty}^{\infty} f(x)dx = 1$ if the pdf is valid.

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^5 kdx = [k \cdot x]_{x=0}^5 = 5k$$

Hence, $k = 1/5$.

b. Sketch a graph of $f(x)$ on the Cartesian plane.



c. Find the cdf $F(x)$ of X.

If $x < 0$, then:

$$0 \leq F(x) = \int_{-\infty}^{x} f(t)dt \leq \int_{-\infty}^{0} f(t)dt = \int_{-\infty}^{0} 0dt = 0$$

And hence, $F(x) = 0$. If $0 \leq x \leq 5$, then:

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{0} f(t)dt + \int_{0}^{x} f(t)dt = 0 + \int_{0}^{x} \frac{t}{5}dt = \frac{x}{5}$$

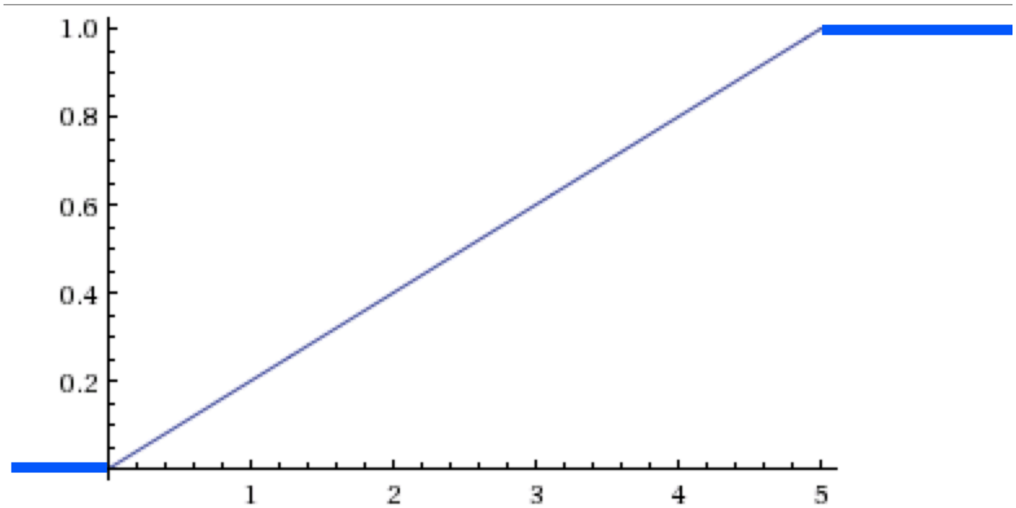
If $x > 5$, then:

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{0} f(t)dt + \int_{0}^{5} f(t)dt + \int_{5}^{x} f(t)dt \\ &= 0 + (\frac{x}{5})_{x=0}^{5} + 0 \\ &= 1 \end{aligned}$$

Thus,

$$F(x) = \begin{cases} 0 & : x < 0 \\ \frac{x}{5} & : 0 \leq x \leq 5 \\ 1 & : x > 5 \end{cases}$$

d. Sketch a graph of $F(x)$ on the Cartesian plane.



e. Find $P(0.2 \leq X \leq 2)$

$$P(0.2 \leq X \leq 2) = F(2) - F(0.2) = \frac{2}{5} - \frac{0.2}{5} = 0.36$$

f. Find $P(X \geq 3)$

$$P(X \geq 3) - 1 - P(X \leq 3) = 1 - F(3) = 1 - 3/5 = 2/5$$

g. Find $P(X = 0.5)$

$P(X = 0.5) = P(X \leq 0.5) - P(X < 0.5) = F(0.5) - F(0.5) = 0$

h. Find $E(X)$

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^5 x \cdot \frac{1}{5} = \frac{1}{10}x^2|_{x=0}^5 = 2.5$$

i. Find $Var(X)$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^5 x^2 \cdot \frac{1}{5} = \frac{1}{15}x^3|_{x=0}^5 = \frac{125}{15} = \frac{25}{3}$$

Hence:

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{25}{3} - (2.5)^2 = 2.0833$$

Exercise 1.3.

a. Find $P(Z \leq 1)$, $Z \sim N(0,1)$

$$P(Z \leq 1) = \Phi(1) = 0.8413$$

from the standard normal table

b. Find $P(0 \leq X \leq 2)$, $X \sim N(3,4)$

$$\begin{aligned}
P(0 \leq X \leq 2) &= P(\frac{0-3}{2} \leq \frac{X-3}{2} \leq \frac{2-3}{2}) \\
&= P(-3/2 \leq Z \leq -1/2) \\
&= P(Z \leq -1/2) - P(Z \leq -3/2) \\
&= \Phi(-0.5) - \Phi(-1.5) \\
&= 0.3085 - 0.0668 \\
&= 0.2417
\end{aligned}$$

c. Find $P(|X-2| < 4)$, $X \sim N(10,3)$

$$\begin{aligned}
P(|X-2| < 4) &= P(-4 < X - 2 < 4) \\
&= P\left(\frac{-4-8}{\sqrt{3}} < \frac{X-2-8}{\sqrt{3}} < \frac{4-8}{\sqrt{3}}\right) \\
&= P\left(\frac{-12}{\sqrt{3}} < \frac{X-10}{\sqrt{3}} < \frac{-4}{\sqrt{3}}\right) \\
&= P(-6.93 < Z < -2.31) \\
&= P(Z < -2.31) - P(Z < -6.93) \\
&= \Phi(-2.31) - \Phi(-6.93) \\
&\approx 0.010 - 0 \\
&= 0.01
\end{aligned}$$

d. Find $P(|X+3|>5)$, $X \sim N(-1,2)$

$$\begin{aligned}
P(X+3>5)+P(X+3<-5) &= P(X>2)+P(X<-8)\\
&= P\left(\frac{X+1}{\sqrt{2}} > \frac{2+1}{\sqrt{2}}\right) + P\left(\frac{X+1}{\sqrt{2}} < \frac{-8+1}{\sqrt{2}}\right)\\
&= P\left(\frac{X-(-1)}{\sqrt{2}} > 2.12\right) + P\left(\frac{X-(-1)}{\sqrt{2}} < -4.95\right)\\
&\approx P(Z>2.12)+P(Z<-4.95)\\
&= 1-P(Z\le 2.12)+P(Z\le -4.95)\\
&\approx 1-0.9830+0\\
&= 0.017
\end{aligned}$$

e. Find the number c such that $P(|Z|>c)=0.85$

$$\begin{aligned}
0.85 &= P(|Z|>c)\\
&= P(Z>c)+P(Z<-c)\\
&= P(Z<-c)+P(Z<-c) \text{ by symmetry}\\
&= 2P(Z<-c)
\end{aligned}$$

Hence:

$$\begin{aligned}
0.425 &= P(Z<-c)\\
\Phi^{-1}(0.425) &= -c\\
-0.19 &= -c\\
0.19 &= c
\end{aligned}$$

f. Find the number c such that $P(|X+2|<c)=0.7$, $X \sim N(-2,9)$

$$\begin{aligned}
0.7 &= P(|X+2|<c) \\
&= P(-c<X+2<c) \\
&= P(-c/3<\frac{X-(-2)}{3}<c/3) \\
&= P(-c/3<Z<c/3) \\
&= P(Z<c/3)-P(Z<-c/3) \\
&= (1-P(Z>c/3))-P(Z<-c/3) \\
&= (1-P(Z<-c/3)-P(Z<-c/3) \\
&= 1-2P(Z<-c/3)
\end{aligned}$$

Hence:

$$\begin{aligned}
0.7 &= 1-2P(Z<-c/3) \\
0.15 &= P(Z<-c/3) \\
\Phi^{-1}(0.15) &= -c/3 \\
-1.04 &= -c/3 \\
3.12 &= c
\end{aligned}$$

g. Find the number c such that $P(|X|>c)=0.6$, $X \sim N(5,3)$

$$\begin{aligned}
0.6 &= P(|X-5|>c) \\
&= P(X-5>c)+P(X-5<-c) \\
&= P(\frac{X-5}{\sqrt{3}}>\frac{c}{\sqrt{3}})+P(\frac{X-5}{\sqrt{3}}<\frac{-c}{\sqrt{3}}) \\
&= P(Z>c/\sqrt{3})+P(Z<-c/\sqrt{3}) \\
&= P(Z<-c/\sqrt{3})+P(Z<-c/\sqrt{3}) \\
&= 2P(Z<-c/\sqrt{3})
\end{aligned}$$

Hence:

$$\begin{aligned}
0.6 &= 2P(Z < -c/\sqrt{3}) \\
0.3 &= P(Z < -c/\sqrt{3}) \\
\Phi^{-1}(0.3) &= -c/\sqrt{3} \\
-0.52 &= -c/\sqrt{3}0.90 = c
\end{aligned}$$

h. Find $t_{5,0.95}$

$t_{5,0.95} = 2.015$ from the t table.

i. Find $t_{7,0.1}$

$t_{7,0.1} = -t_{7,0.9}$ (by symmetry) $= -1.415$ from the t table

j. Find $\chi^2_{2,0.95}$

$\chi^2_{2,0.95} = 5.991$ from the chi-square table.

k. Find $\chi^2_{5,0.9}$

$\chi^2_{5,0.9} = 9.236$ from the chi-square table.

l. Find $F_{5,6,0.9}$

$F_{5,6,0.9} = 3.108$ from the F table.

m. Find $F_{4,3,0.99}$

$F_{4,3,0.99} = 28.710$ from the F table.

Exercise 1.4.
Use the Central Limit Theorem to approximate the following:

a. $P(|\overline{X}-1|<2)$, where $X_1, X_2, \ldots, X_{41}$ are iid Exponential(4), each with mean 0.25 and variance 0.0625.

By the Central Limit Theorem, $\overline{X} \sim$ approx. $N(0.25, 0.0625/41) = N(0.25, 0.0015)$. Hence:

$$\begin{aligned}
P(|\overline{X}-1|<2) &= P(-2<\overline{X}-1<2) \\
&= P(\frac{-2+0.75}{\sqrt{0.0015}} < \frac{\overline{x}-0.25}{\sqrt{0.0015}} < \frac{2+0.75}{\sqrt{0.0015}}) \\
&\approx P(-32.27 < Z < 71.00) \\
&= P(Z<71.00) - P(Z<-32.27) \\
&\approx 1-0 \\
&= 1
\end{aligned}$$

b. The number c such that $P(\overline{X}>c) = 0.95$, where $X_1, X_2, \ldots, X_{26}$ are iid Gamma(1,2), each with mean 2 and variance 4.

By the Central Limit Theorem, $\overline{X} \sim$ approx. $N(2, 4/26) = N(2, 0.154)$.

$$\begin{aligned}
0.95 &= P(\overline{X}>c) \\
&= P(\frac{\overline{X}-2}{\sqrt{0.154}} > \frac{c-2}{\sqrt{0.154}}) \\
&\approx P(Z > \frac{c-2}{\sqrt{0.154}})
\end{aligned}$$

Hence:

$$
\begin{aligned}
0.95 &= P(Z > \frac{c-2}{\sqrt{0.154}}) \\
0.95 &= 1 - P(Z \le \frac{c-2}{\sqrt{0.154}}) \\
0.05 &= P(Z \le \frac{c-2}{\sqrt{0.154}}) \\
0.05 &= \Phi(\frac{c-2}{\sqrt{0.154}}) \\
\Phi^{-1}(0.05) &= \frac{c-2}{\sqrt{0.154}} \\
-1.64 &= \frac{c-2}{\sqrt{0.154}} \\
1.36 &= c
\end{aligned}
$$

c. $P(|\overline{X} - 5| > 1)$, where $X_1, X_2, \ldots, X_{38} \sim$ are iid χ^2_5, each with mean 5 and variance 10.

By the Central Limit Theorem, $\overline{X} \sim$ approx. $N(5, 10/38) = N(5, 0.263)$

$$
\begin{aligned}
P(|\overline{X} - 5| > 1) &= P(\overline{X} - 5 > 1) + P(\overline{X} - 5 < -1) \\
&= P(\frac{\overline{X} - 5}{0.263} > \frac{1}{0.263}) + P(\frac{\overline{X} - 5}{0.263} < \frac{-1}{0.263}) \\
&\approx P(Z > 3.80) + P(Z < -3.80) \\
&\approx 0 + 0 \\
&= 0
\end{aligned}
$$

Exercise 1.5.