6 Introduction to formal statistical inference

Formal statistical inference uses probability theory to quantify the reliability of data-based conclusions. We want information on a population. We can use:

1. Point estimates:

2. Interval estimates:

6.1 Large-sample confidence intervals for a mean

Many important engineering applications of statistics fit the following mold. Values for parameters of a data-generating process are unknown. Based on data, the goal is

1.

2.

Definition 6.1. A *confidence interval* for a parameter (or function of one or more parameters) is a data-based interval of numbers thought likely to contain the parameter (or function of one or more parameters) possessing a stated probability-based confidence or reliability.

A confidence interval is a realization of a **random interval**, an interval on the real line with a random variable at one or both of the endpoints.

Example 6.1 (Instrumental drift)**.** Let Z be a measure of instrumental drift of a random voltmeter that comes out of a certain factory. Say $Z \sim N(0,1)$. Define a random interval:

$$(Z - 2, Z + 2)$$

What is the probability that -1 is inside the interval?

Example 6.2 (More practice)**.** Calculate:

1. $P(2 \text{ in } (X - 1, X + 1))$, $X \sim N(2, 4)$

2. $P(6.6 \text{ in } (X - 2, X + 1))$, $X \sim N(7, 2)$

Example 6.3 (Abstract random intervals)**.** Let's say $X_1, X_2, \ldots, X_n$ are iid with $n \geq 25$, mean μ, variance σ^2. We can find a random interval that provides a lower bound for μ with $1 - \alpha$ probability:

Calculate:

1. $P(\mu \in (-\infty, \overline{X} + z_{1-\alpha}\frac{\sigma}{\sqrt{n}})), X \sim N(\mu, \sigma^2)$

2. $P(\mu \in (\overline{X} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{X} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}})), X \sim N(\mu, \sigma^2)$

6.1.1 A Large-n confidence interval for μ involving σ

A $1 - \alpha$ **confidence interval** for an unknown parameter is the realization of a random interval that contains that parameter with probability $1 - \alpha$.

For random variables $X_1, X_2, \ldots, X_n$ iid with $\mathrm{E}(X_1) = \mu$, $\mathrm{Var}(X_1) = \sigma^2$, a $1 - \alpha$ confidence interval for μ is

$$\left(\overline{x} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{x} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

which is a **realization** from the random interval

$$\left(\overline{X} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{X} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right).$$

- Two-sided $1 - \alpha$ confidence interval for μ

- One-sided $1 - \alpha$ confidence interval for μ with a upper confidence bound

- One-sided $1 - \alpha$ confidence interval for μ with a lower confidence bound

Example 6.4 (Fill weight of jars)**.** Suppose a manufacturer fills jars of food using a stable filling process with a known standard deviation of $\sigma = 1.6$g. We take a sample of $n = 47$ jars and measure the sample mean weight $\bar{x} = 138.2$g. A two-sided 90% confidence interval $(\alpha = 0.1)$ for the true mean weight μ is:

Interpretation:

What if we just want to be sure that the true mean fill weight is high enough?

Example 6.5 (Hard disk failures)**.** F. Willett, in the article "The Case of the Derailed Disk Drives?" (*Mechanical Engineering*, 1988), discusses a study done to isolate the cause of link code A failure in a model of Winchester hard disk drive. For each disk, the investigator measured the breakaway torque (in. oz.) required to loosen the drive's interrupter flag on the stepper motor shaft. Breakaway torques for 26 disk drives were recorded, with a sample mean of 11.5 in. oz. Suppose you know the true standard deviation of the breakaway torques is 5.1 in. oz. Calculate and interpret:

1. A two-sided 90% confidence interval for the true mean breakaway torque of the relevant type of Winchester drive.

2. An analogous two-sided 95% confidence interval.

Example 6.6 (Width of a CI)**.** If you want to estimate the breakaway torque with a 2-sided, 95% confidence interval with ± 2.0 in. oz. of precision, what sample size would you need?

6.1.2 A generally applicable large-n confidence interval for μ

Although the equations for a $1 - \alpha$ confidence interval is mathematically correct, it is severely limited in its usefulness because

If $n \geq 25$ and σ is *unknown*, $Z = \frac{\overline{X} - \mu}{s/\sqrt{n}}$, where

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \overline{x})^2}.$$

is still **approximately standard normally distributed**. So, you can replace σ in the confidence interval formula with the sample standard deviation, s.

- Two-sided $1 - \alpha$ confidence interval for μ

- One-sided $1 - \alpha$ confidence interval for μ with a upper confidence bound

- One-sided $1 - \alpha$ confidence interval for μ with a lower confidence bound

Example 6.7. Suppose you are a manufacturer of construction equipment. You make 0.0125 inch wire rope and need to determine how much weight it can hold before breaking so that you can label it clearly. Here are breaking strengths, in kg, for 41 sample wires:

```
 [1] 100.37  96.31  72.57  88.02 105.89 107.80  75.84  92.73  67.47  94.87
[11] 122.04 115.12  95.24 119.75 114.83 101.79  80.90  96.10 118.51 109.66
[21]  88.07  56.29  86.50  57.62  74.70  92.53  86.25  82.56  97.96  94.92
[31]  62.00  93.00  98.44 119.37 103.70  72.40  71.29 107.24  64.82  93.51
[41]  86.97
```

The sample mean breaking strength is 91.85 kg and the sample standard deviation is 17.6 kg. Using the appropriate 95% confidence interval, try to determine whether the breaking strengths meet the requirement of at least 85 kg.

6.2   Small-sample confidence intervals for a mean

The most important practical limitation on the use of the methods of the previous sections is

That restriction comes from the fact that without it,

So, if one mechanically uses the large-n interval formula $\bar{x} \pm z \frac{s}{\sqrt{n}}$ with a small sample,

If it is sensible to model the observations as iid normal random variables, then we can arrive at inference methods for small-n sample means.

6.2.1 The Student t distribution

Definition 6.2. The *(Student) t distribution with degrees of freedom parameter ν* is a continuous probability distribution with probability density

$$f(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}} \left(1 + \frac{t^2}{\nu}\right)^{-(\nu+1)/2} \qquad \text{for all } t.$$

The t distribution

- is bell-shaped and symmetric about 0
- has fatter tails than the normal, but approaches the shape of the normal as $\nu \to \infty$.

We use the t table (Table B.4 in Vardeman and Jobe) to calculate quantiles.
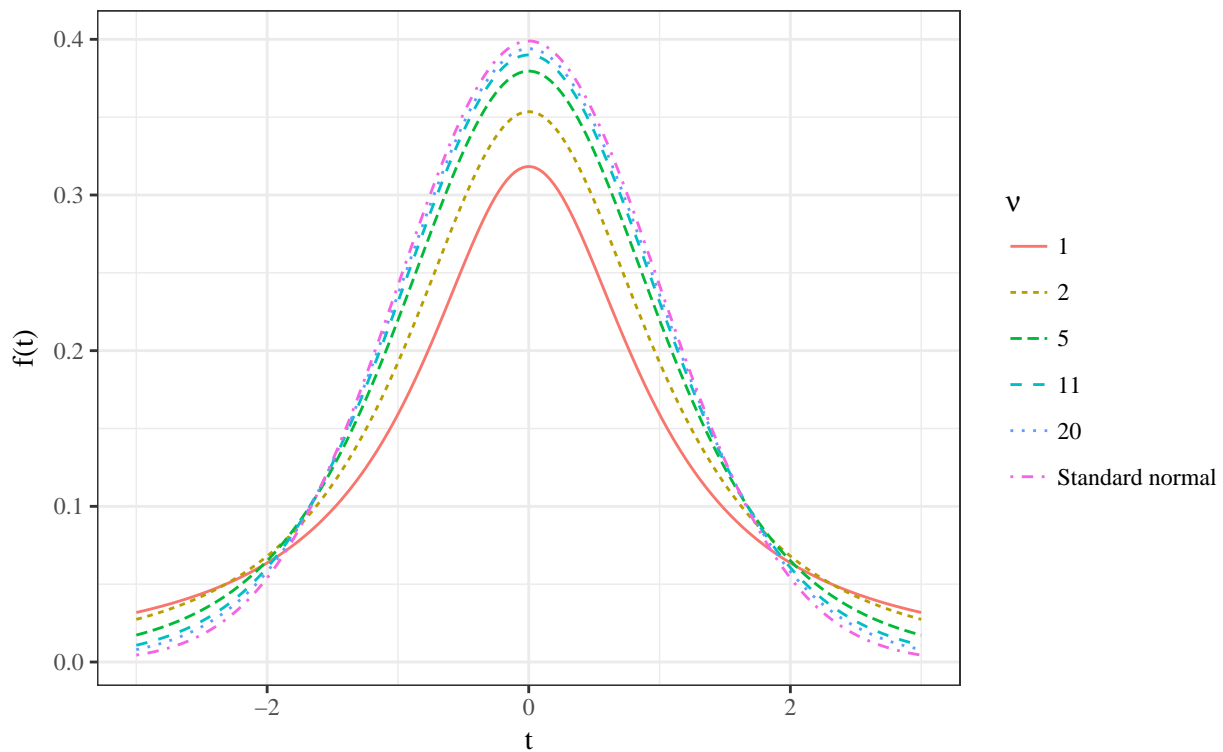
Example 6.8 (t quantiles). Say $T \sim t_5$. Find c such that $P(T \leq c) = 0.9$.

Table B.4
t Distribution Quantiles

ν	$Q(.9)$	$Q(.95)$	$Q(.975)$	$Q(.99)$	$Q(.995)$	$Q(.999)$	$Q(.9995)$
1	3.078	6.314	12.706	31.821	63.657	318.317	636.607
2	1.886	2.920	4.303	6.965	9.925	22.327	31.598
3	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	1.476	2.015	2.571	3.365	4.032	5.893	6.869

Figure 1: Student's t distribution quantiles.

6.2.2 Small-sample confidence intervals, σ unknown

If we can assume that $X_1, \ldots, X_n$ are iid with mean μ and variance σ^2, and are also normally distributed,

We can then use $t_{n-1,1-\alpha/2}$ instead of $z_{1-\alpha/2}$ in the confidence intervals.

- Two-sided $1 - \alpha$ confidence interval for μ

- One-sided $1 - \alpha$ confidence interval for μ with a upper confidence bound

- One-sided $1 - \alpha$ confidence interval for μ with a lower confidence bound

Example 6.9 (Concrete beams)**.** 10 concrete beams were each measured for flexural strength (MPa). Assuming the flexural strengths are iid normal, calculate and interpret a two-sided 99% CI for the flexural strength of the beams.

[1] 8.2 8.7 7.8 9.7 7.4 7.8 7.7 11.6 11.3 11.8

Is the true mean flexural strength below the minimum requirement of 11 MPa? Find out with the appropriate 95% CI.

Example 6.10 (Paint thickness). Consider the following sample of observations on coating thickness for low-viscosity paint.

```
 [1] 0.83 0.88 0.88 1.04 1.09 1.12 1.29 1.31 1.48 1.49 1.59 1.62 1.65 1.71
[15] 1.76 1.83
```

A normal QQ plot shows that they are close enough to normally distributed.
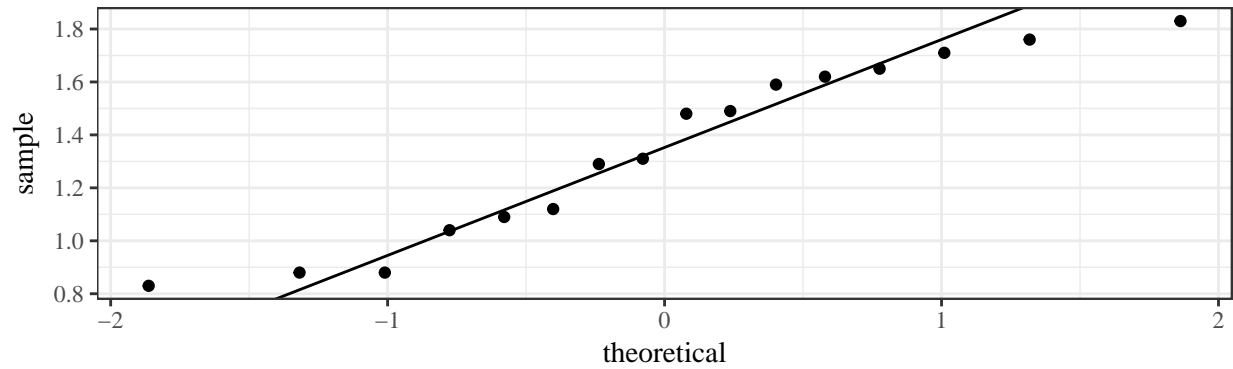
Calculate and interpret a two-sided 90% confidence interval for the true mean thickness.

6.3 Hypothesis testing

Last section illustrated how probability can enable confidence interval estimation. We can also use probability as a means to use data to quantitatively assess the plausibility of a trial value of a parameter.

Statistical inference is using data from the sample to draw conclusions about the population.

1. Interval estimation (confidence intervals)

2. Hypothesis testing

Definition 6.3. Statistical *significance testing* is the use of data in th quantitative assessment of the plausibility of some trial value for a parameter (or function of one or more parameters).

Significance (or hypothesis) testing begins with the specification of a trial value (or **hypothesis**).

Definition 6.4. A *null hypothesis* is a statement of the form

$$\text{Parameter} = \#$$

or

$$\text{Function of parameters} = \#$$

for some # that forms the basis of investigation in a significance test. A null hypothesis is usually formed to embody a status quo/"pre-data" view of the parameter. It is denoted H_0.

Definition 6.5. An *alternative hypothesis* is a statement that stands in opposition to the null hypothesis. It specifies what forms of departure from the null hypothesis are of concern. An alternative hypothesis is denoted as H_a. It is of the form

$$\text{Parameter} \neq \# \quad \text{or} \quad \text{Parameter} > \# \quad \text{or} \quad \text{Parameter} < \#$$

Examples (testing the true mean value):

$$H_0 : \mu = \# \quad H_0 : \mu = \# \quad H_0 : \mu = \#$$
$$H_a : \mu \neq \# \quad H_a : \mu > \# \quad H_a : \mu < \#$$

Often, the alternative hypothesis is based on an investigator's suspicions and/or hopes about th true state of affairs.

The **goal** is to use the data to debunk the null hypothesis in favor of the alternative.

1. Assume H_0.
2. Try to show that, under H_0, the data are preposterous.
3. If the data are preposterous, reject H_0 and conclude H_a.

The outcomes of a hypothesis test consists of:

Example 6.11 (Fair coin). Suppose we toss a coin $n = 25$ times, and the results are denoted by $X_1, X_2, \ldots, X_{25}$. We use 1 to denote the result of a head and 0 to denote the results of a tail. Then $X_1 \sim Binomial(1, \rho)$ where ρ denotes the chance of getting heads, so $\mathrm{E}(X_1) = \rho, \mathrm{Var}(X_1) = \rho(1 - \rho)$. Given the result is you got all heads, do you think the coin is fair?

In the real life, we may have data from many different kinds of distributions! Thus we need a universal framework to deal with these kinds of problems.

6.3.1 Significance tests for a mean

Definition 6.6. A *test statistic* is the particular form of numerical data summarization used in a significance test.

Definition 6.7. A *reference (or null) distribution* for a test statistic is the probability distribution describing the test statistic, provided the null hypothesis is in fact true.

Definition 6.8. The *observed level of significance or p-value* in a significance test is the probability that the reference distribution assigns to the set of possible values of the test statistic that are at least as extreme as the one actually observed.

Based on our results from Section 6.2 of the notes, we can develop hypothesis tests for the true mean value of a distribution in various situations, given an iid sample $X_1, \ldots, X_n$ where $H_0 : \mu = \mu_0$.

Let K be the value of the test statistic, $Z \sim N(0, 1)$, and $T \sim t_{n-1}$. Here is a table of p-values that you should use for each set of conditions and choice of H_a.

Situation	K	$H_a : \mu \neq \mu_0$	$H_a : \mu < \mu_0$	$H_a : \mu > \mu_0$
$n \geq 25, \sigma$ known	$\frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$	$P(|Z| > K)$	$P(Z < K)$	$P(Z > K)$
$n \geq 25, \sigma$ unknown	$\frac{\bar{x} - \mu_0}{s/\sqrt{n}}$	$P(|Z| > K)$	$P(Z < K)$	$P(Z > K)$
$n < 25, \sigma$ unknown	$\frac{\bar{x} - \mu_0}{s/\sqrt{n}}$	$P(|T| > K)$	$P(T < K)$	$P(T > K)$

Steps to perform a hypothesis test:

1.

2.

3.

4.

5.

6.

Example 6.12 (Cylinders)**.** The strengths of 40 steel cylinders were measured in MPa. The sample mean strength is 1.2 MPa with a sample standard deviation of 0.5 MPa. At significance level $\alpha = 0.01$, conduct a hypothesis test to determine if the cylinders meet the strength requirement of 0.8 MPa.

Example 6.13 (Concrete beams)**.** 10 concrete beams were each measured for flexural strength (MPa). The data is as follows.

[1] 8.2 8.7 7.8 9.7 7.4 7.8 7.7 11.6 11.3 11.8

The sample mean was 9.2 MPa and the sample variance was 3.0933 MPa. Conduct a hypothesis test to find out if the flexural strength is different from 9.0 MPa.

6.3.2 Hypothesis testing using the CI

We can also use the $1 - \alpha$ confidence interval to perform hypothesis tests (instead of p-values). The confidence interval will contain μ_0 when there is little to no evidence against H_0 and will not contain μ_0 when there is strong evidence against H_0.

Steps to perform a hypothesis test using a confidence interval:

1.

2.

3.

4.

5.

6.

Example 6.14 (Breaking strength of wire, cont'd)**.** Suppose you are a manufacturer of construction equipment. You make 0.0125 inch wire rope and need to determine how much weight it can hold before breaking so that you can label it clearly. You have breaking strengths, in kg, for 41 sample wires with sample mean breaking strength 91.85 kg and sample standard deviation 17.6 kg. Using the appropriate 95% confidence interval, conduct a hypothesis test to find out if the true mean breaking strength is above 85 kg.

Example 6.15 (Concrete beams, cont'd)**.** 10 concrete beams were each measured for flexural strength (MPa). The data is as follows.

[1] 8.2 8.7 7.8 9.7 7.4 7.8 7.7 11.6 11.3 11.8

The sample mean was 9.2 MPa and the sample variance was 3.0933 MPa. At $\alpha = 0.01$, test the hypothesis that the true mean flexural strength is 10 MPa using a confidence interval.

Example 6.16 (Paint thickness, cont'd)**.** Consider the following sample of observations on coating thickness for low-viscosity paint.

[1] 0.83 0.88 0.88 1.04 1.09 1.12 1.29 1.31 1.48 1.49 1.59 1.62 1.65 1.71 [15] 1.76 1.83

Using $\alpha = 0.1$, test the hypothesis that the true mean paint thickness is 1.00 mm. Note, the 90% confidence interval for the true mean paint thickness was calculated from before as $(1.201, 1.499)$.

6.4 Inference for matched pairs and two-sample data

An important type of application of confidence interval estimation and significance testing is when we either have *paired data* or *two-sample* data.

6.4.1 Matched pairs

Recall,

Examples:

One simple method of investigating the possibility of a consistent difference between paired data is to

1.

2.

Example 6.17 (Fuel economy). Twelve cars were equipped with radial tires and driven over a test course. Then the same twelve cars (with the same drivers) were equipped with regular belted tires and driven over the same course. After each run, the cars gas economy (in km/l) was measured. Using significance level $\alpha = 0.05$ and the method of critical values, test for a difference in fuel economy between the radial tires and belted tires. Construct a 95% confidence interval for true mean difference due to tire type.

car	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	11.0	12.0
radial	4.2	4.7	6.6	7.0	6.7	4.5	5.7	6.0	7.4	4.9	6.1	5.2
belted	4.1	4.9	6.2	6.9	6.8	4.4	5.7	5.8	6.9	4.7	6.0	4.9

Example 6.18 (End-cut router). Consider the operation of an end-cut router in the manufacture of a company's wood product. Both a leading-edge and a trailing-edge measurement were made on each wooden piece to come off the router. Is the leading-edge measurement different from the trailing-edge measurement for a typical wood piece? Do a hypothesis test at $\alpha = 0.05$ to find out. Make a two-sided 95% confidence interval for the true mean of the difference between the measurements.

piece	1.000	2.000	3.000	4.000	5.000
leading_edge	0.168	0.170	0.165	0.165	0.170
trailing_edge	0.169	0.168	0.168	0.168	0.169

6.4.2   Two-sample data

Paired differences provide inference methods of a special kind for comparison. Methods that can be used to compare two means where two different *unrelated* samples will be discussed next.

Examples:

Notation:

6.4.2.1 Large samples ($n_1 \geq 25, n_2 \geq 25$)

The difference in sample means $\bar{x}_1 - \bar{x}_2$ is a natural statistic to use in comparing μ_1 and μ_2.

If σ_1 and σ_2 are **known**, then Proposition 5.1 tells us

$$\mathrm{E}(\overline{X}_1 - \overline{X}_2) =$$

$$\mathrm{Var}(\overline{X}_1 - \overline{X}_2) =$$

If, in addition, n_1 and n_2 are large,

So, if we want to test $H_0 : \mu_1 - \mu_2 = \#$ with some alternative hypothesis, σ_1 and σ_2 are known, and $n_1 \geq 25, n_2 \geq 25$, then we use the statistic

$K =$

which has a $N(0, 1)$ distribution if

1. H_0 is true
2. The sample 1 points are iid with mean μ_1 and variance σ_1^2, and the sample 2 points are iid with mean μ_2 and variance σ_2^2.

The confidence intervals (2-sided, 1-sided upper, and 1-sided lower, respectively) for $\mu_1 - \mu_2$ are:

If σ_1 and σ_2 are **unknown**, and $n_1 \geq 25, n_2 \geq 25$, then we use the statistic

$$K =$$

and confidence intervals (2-sided, 1-sided upper, and 1-sided lower, respectively) for $\mu_1 - \mu_2$:

Example 6.19 (Anchor bolts). An experiment carried out to study various characteristics of anchor bolts resulted in 78 observations on shear strength (kip) of 3/8-in. diameter bolts and 88 observations on strength of 1/2-in. diameter bolts. Let Sample 1 be the 1/2 in diameter bolts and Sample 2 be the 3/8 indiameter bolts. Using a significance level of $\alpha = 0.01$, find out if the 1/2 in bolts are more than 2 kip stronger (in shear strength) than the 3/8 in bolts. Calculate and interpret the appropriate 99% confidence interval to support the analysis.

- $n1 = 88, n2 = 78$
- $\bar{x}_1 = 7.14, \bar{x}_2 = 4.25$
- $s_1 = 1.68, s_2 = 1.3$

6.4.2.2 Small samples

If $n_1 < 25$ or $n_2 < 25$, then we need some **other assumptions** to hold in order to complete inference on two-sample data.

A test statistic to test $H_0 : \mu_1 - \mu_2 = \#$ against some alternative is $K =$

Also assuming - H_0 is true, - The sample 1 points are iid $N(\mu_1, \sigma_1^2)$, the sample 2 points are iid $N(\mu_2, \sigma_2^2)$, - and the sample 1 points are independent of the sample 2 points.

Then $K \sim$

$1 - \alpha$ confidence intervals (2-sided, 1-sided upper, and 1-sided lower, respectively) for $\mu_1 - \mu_2$ under these assumptions are of the form:

Example 6.20 (Springs)**.** The data of W. Armstrong on spring lifetimes (appearing in the book by Cox and Oakes) not only concern spring longevity at a 950 N/mm^2 stress level but also longevity at a 900 N/mm^2 stress level. Let sample 1 be the 900 N/mm^2 stress group and sample 2 be the 950 N/mm^2 stress group. Let's do a hypothesis test to see if the sample 1 springs lasted significantly longer than the sample 2 springs.

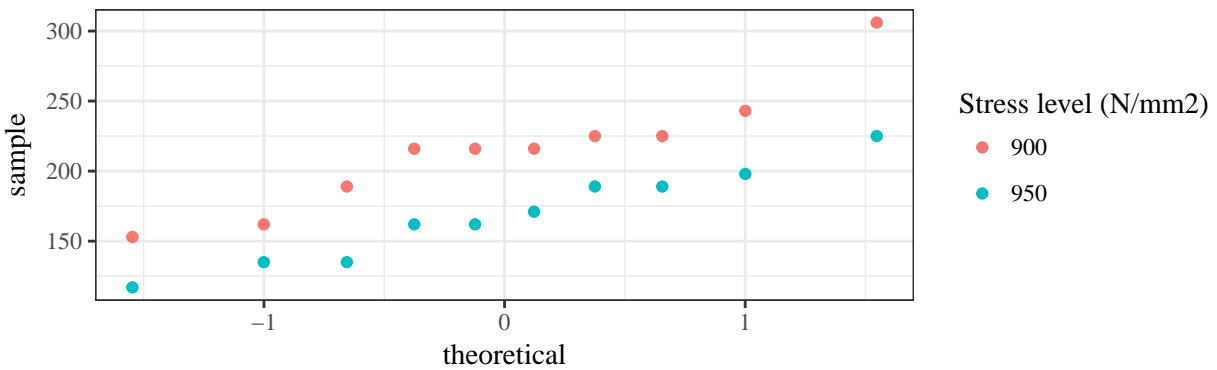| 900 N/mm2 Stress	950 N/mm2 Stress
216, 162, 153, 216, 225, 216, 306, 225, 243, 189	225, 171, 198, 189, 189, 135, 162, 135, 117, 162

Figure 2: Normal plots of spring lifetimes under two different levels of stress.

Example 6.21 (Stopping distance)**.** Suppose μ_1 and μ_2 are true mean stopping distances (in meters) at 50 mph for cars of a certain type equipped with two different types of breaking systems. Suppose $n_1 = n_2 = 6$, $\bar{x}_1 = 115.7$, $\bar{x}_2 = 129.3$, $s_1 = 5.08$, and $s_2 = 5.38$. Use significance level $\alpha = 0.01$ to test $H_0 : \mu_1 - \mu_2 = -10$ vs. $H_A : \mu_1 - \mu_2 < -10$. Construct a 2-sided 99

6.5 Prediction intervals

Methods of confidence interval estimation andd hypothesis testing concern the problem of reasoning from sample information to statements about underlying *parameters* of the data generation (such as μ).

Sometimes it is useful to not make a statement about a parameter value, but create bounds on other *individual values* generated by the process.

> How can we use out data $x_1, \ldots, x_n$ to create an interval likely to contain one additional (as yet unobserved) value x_{n+1} from the same data generating mechanism?

Let $X_1, \ldots, X_n$ be iid Normal random variables with

$$\mathrm{E}(X_i) = \mu \text{ for all } i = 1, \ldots, n$$
$$\mathrm{Var}(X_i) = \sigma^2 \text{ for all } i = 1, \ldots, n$$

Then,

Let X_{n+1} be an additional observation from the same data generating mechanism.

$\mathrm{E}(\overline{X}_n - X_{n+1}) =$

$\mathrm{Var}(\overline{X}_n - X_{n+1}) =$

So,

Generally, σ is unknown, so replace σ by s, and

Then, $1 - \alpha$ **Prediction intervals** for X_{n+1} are

Table B.4
t Distribution Quantiles

ν	$Q(.9)$	$Q(.95)$	$Q(.975)$	$Q(.99)$	$Q(.995)$	$Q(.999)$	$Q(.9995)$
1	3.078	6.314	12.706	31.821	63.657	318.317	636.607
2	1.886	2.920	4.303	6.965	9.925	22.327	31.598
3	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	1.325	1.725	2.086	2.528	2.845	3.552	3.849
21	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	1.319	1.714	2.069	2.500	2.807	3.485	3.768
24	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	1.296	1.671	2.000	2.390	2.660	3.232	3.460
120	1.289	1.658	1.980	2.358	2.617	3.160	3.373
∞	1.282	1.645	1.960	2.326	2.576	3.090	3.291

This table was generated using MINITAB.