5 Probability: the mathematics of randomness

The theory of probability is the mathematician's description of random variation. This chapter introduces enough probability to serve as a minimum background for making formal statistical inferences.

5.1 (Discrete) random variables

The concept of a random variable is introduced in general terms and the special case of discrete data is considered.

5.1.1 Random variables and distributions

It is helpful to think of data values as subject to chance influences. Chance is commonly introduced into the data collection process through

1.

2.

3.

Definition 5.1. A *random variable* is a quantity that (prior to observation) can be thought of as dependent on chance phenomena.

Definition 5.2. A *discrete random variable* is one that has isolated or separated possible values (rather than a continuum of available outcomes).

Definition 5.3. A *continuous random variable* is one that can be idealized as having an entire (continuous) interval of numbers as its set of values.

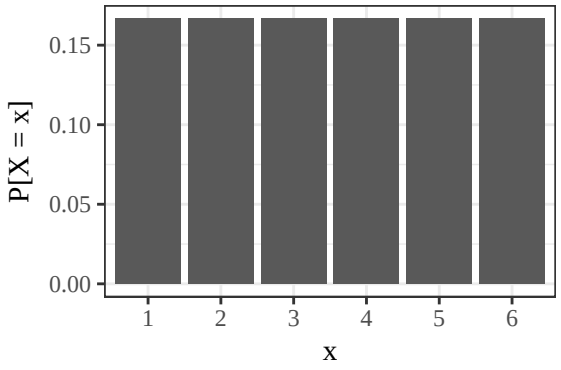
Example 5.1 (Roll of a die)**.**

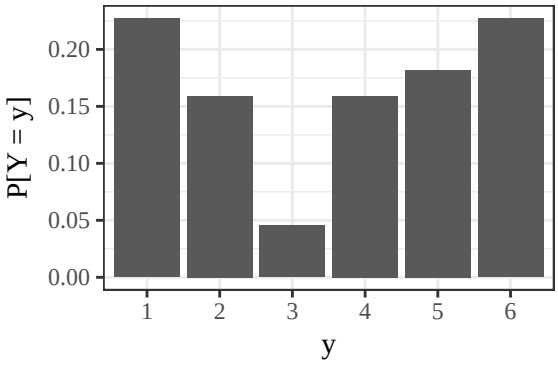
Definition 5.4. To specify a *probability distribution* for a random variable is to give its set of possible values and (in one way or another) consistently assign numbers between 0 and 1 - called *probabilities* - as measures of the likelihood that the various numerical values will occur

Example 5.2 (Roll of a die, cont'd)**.**

x	1	2	3	4	5	6
$P[X = x]$	1/6	1/6	1/6	1/6	1/6	1/6

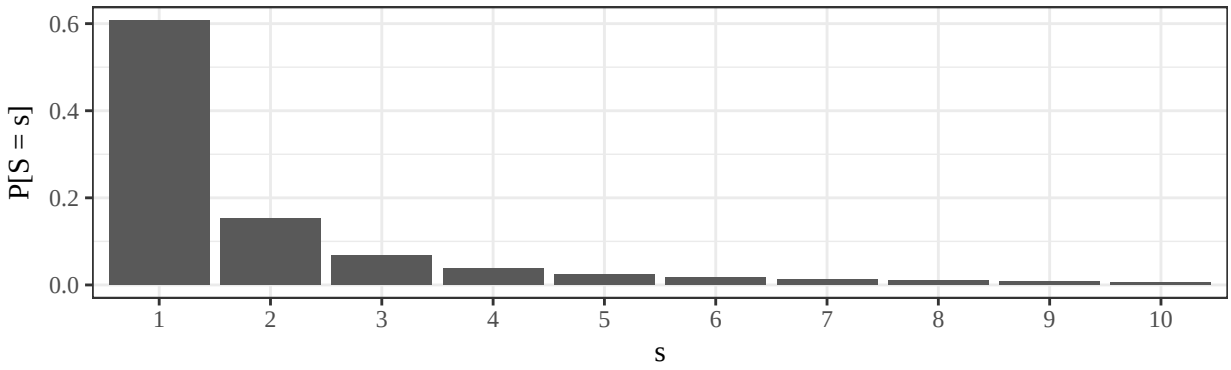
y	1	2	3	4	5	6
$P[Y = y]$	5/22	7/44	1/22	7/44	2/11	5/22





Example 5.3 (Shark attacks)**.** Suppose S is the number of provoked shark attacks off FL next year. This has an infinite number of possible values. Here is one possible (made up) distribution:

s	1	2	3	$\cdots$	k	$\cdots$
$P[S=s]$	$\frac{6}{\pi^2}$	$\frac{1}{2^2}\frac{6}{\pi^2}$	$\frac{1}{3^2}\frac{6}{\pi^2}$	$\cdots$	$\frac{1}{k^2}\frac{6}{\pi^2}$	$\cdots$



5.1.2 Probability mass functions and cumulative distribution functions

The tool most often used to describe a discrete probability distribution is the *probability mass function*.

Definition 5.5. A *probability mass function (pmf)* for a discrete random variable X, having possible values $x_1, x_2, \ldots$, is a non-negative function $f(x)$ with $f(x_1) = P[X = x_1]$, the probability that X takes the value x_1.

Properties of a mathematically valid probability mass function:

1.

2.

A probability mass function $f(x)$ gives probabilities of occurrence for individual values. Adding the appropriate values gives probabilities associated with the occurrence of multiple values.

Example 5.4 (Torque)**.** Let Z = the torque, rounded to the nearest integer, required to loosen the next bolt on an apparatus.

z	11	12	13	14	15	16	17	18	19	20
$f(z)$	0.03	0.03	0.03	0.06	0.26	0.09	0.12	0.20	0.15	0.03

Calculate the following probabilities:

$P(Z \leq 14)$

$P(Z > 16)$

$P(Z \text{ is even})$

$P(Z \text{ in } \{15, 16, 18\})$

Another way of specifying a discrete probability distribution is sometimes used.

Definition 5.6. The *cumulative probability distribution (cdf)* for a random variable X is a function $F(x)$ that for each number x gives the probability that X takes that value or a smaller one, $F(x) = P[X \leq x]$.

Since (for discrete distributions) probabilities are calculated by summing values of $f(x)$,

$$F(x) = P[X \leq x] = \sum_{y \leq x} f(y)$$

Properties of a mathematically valid cumulative distribution function:

1.

2.

3.

4.

Example 5.5 (Torque, cont'd)**.** Let Z = the torque, rounded to the nearest integer, required to loosen the next bolt on an apparatus.

z	11	12	13	14	15	16	17	18	19	20
$F(z)$	0.03	0.06	0.09	0.15	0.41	0.50	0.62	0.82	0.97	1

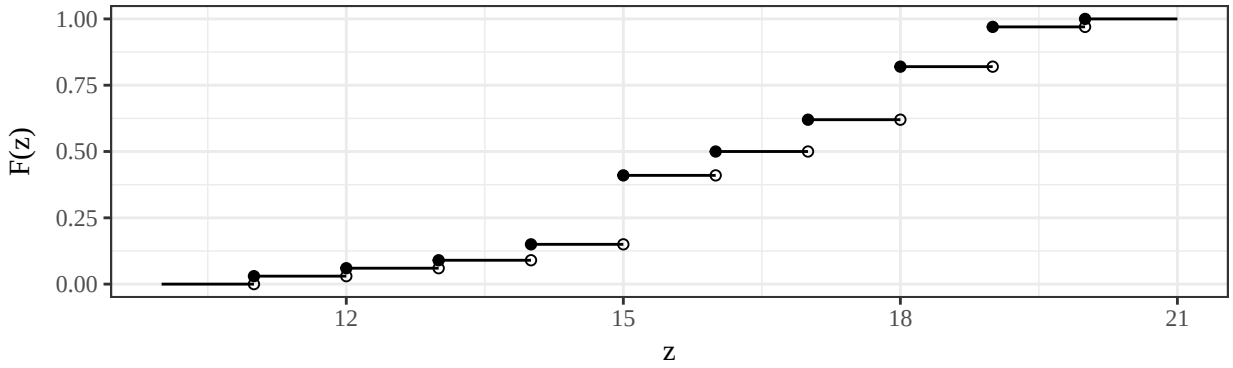


Figure 1: Cdf function for torques.

Calculate the following probabilities using the **cdf only**:

$F(10.7)$

$P(Z \leq 15.5)$

$P(12.1 < Z \leq 14)$

$P(15 \leq Z < 18)$

Example 5.6. Say we have a random variable Q with pmf:

q	$f(q)$
1	0.34
2	0.1
3	0.22
7	0.34

Draw the cdf.

5.1.3 Summaries

Almost all of the devices for describing relative frequency (empirical) distributions in Ch. 3 have versions that can describe (theoretical) probability distributions.

1.

2.

3.

Definition 5.7. The *mean* or *expected value* of a discrete random variable X is

$$\mathrm{E}X = \sum_x x f(x)$$

Example 5.7 (Roll of a die, cont'd)**.** Calculate the expected value of a toss of a fair and unfair die.

x	1	2	3	4	5	6
$P[X = x]$	1/6	1/6	1/6	1/6	1/6	1/6

y	1	2	3	4	5	6
$P[Y = y]$	5/22	7/44	1/22	7/44	2/11	5/22

Example 5.8 (Torque, cont'd)**.** Let Z = the torque, rounded to the nearest integer, required to loosen the next bolt on an apparatus.

z	11	12	13	14	15	16	17	18	19	20
$f(z)$	0.03	0.03	0.03	0.06	0.26	0.09	0.12	0.20	0.15	0.03

Calculate the expected torque required to loosen the next bolt.

Definition 5.8. The *variance* of a discrete random variable X is

$$\mathrm{Var}X = \sum_x (x - \mathrm{E}X)^2 f(x) = \sum_x x^2 f(x) - (\mathrm{E}X)^2.$$

The *standard deviation* of X is $\sqrt{\mathrm{Var}X}$.

Example 5.9. Say we have a random variable Q with pmf:

q	$f(q)$
1	0.34
2	0.1
3	0.22
7	0.34

Calculate the variance and the standard deviation.

Example 5.10 (Roll of a die, cont'd)**.** Calculate the variance and standard deviation of a roll of a fair die.

5.1.4 Special discrete distributions

Discrete probability distributions are sometimes developed from past experience with a particular physical phenomenon.

On the other hand, sometimes an easily manipulated set of mathematical assumptions having the potential to describe a variety of real situations can be put together.

One set of assumptions is that of independent identical success-failure trials where

1.

2.

Consider a variable

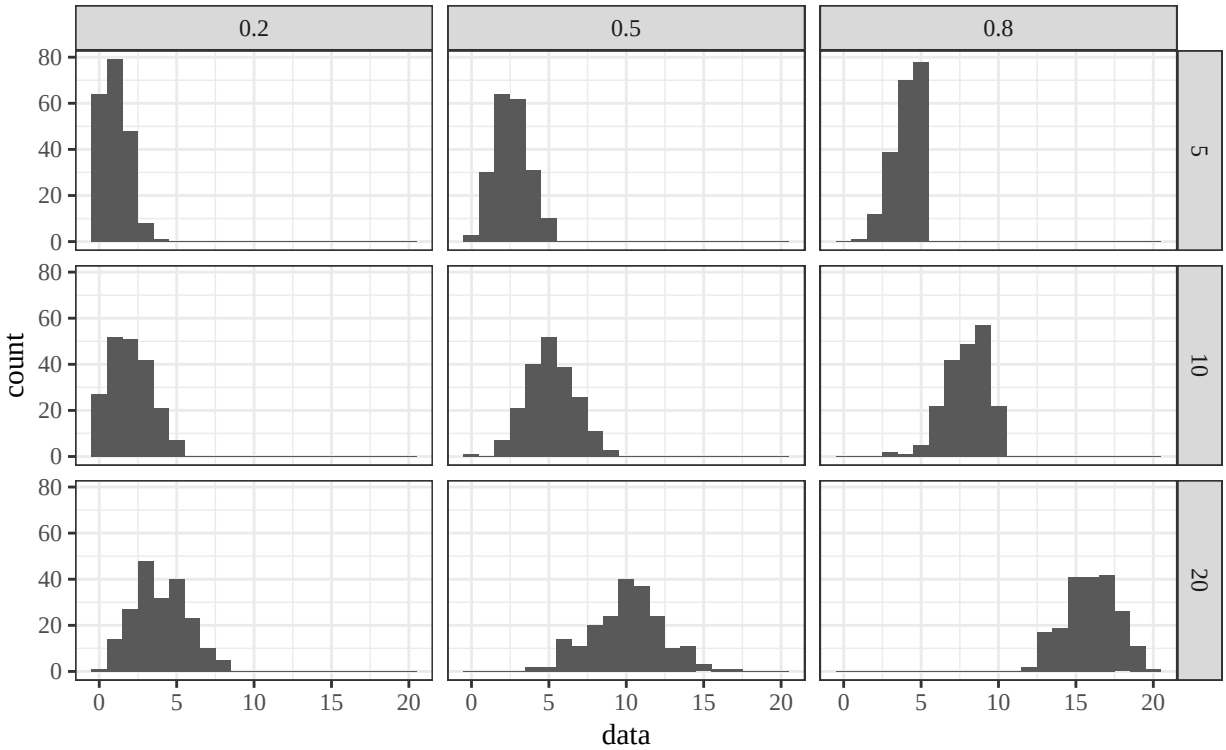
$$X = \text{ the number of successes in } n \text{ independent identicall success-failure trials}$$

Definition 5.9. The *binomial(n, p) distribution* is a discrete probability distribution with pmf

$$f(x) = \begin{cases} \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} & x = 0, 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

for n a positive integer and $0 < p < 1$.

Examples that could follow a binomial(n, p) distribution:



For X a binomial(n, p) random variable,

$$\mu = \mathrm{E}X = \sum_{x=0}^{n} x \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}) = np$$
$$\sigma^2 = \mathrm{Var}X = \sum_{x=0}^{n} (x-np)^2 \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}) = np(1-p)$$

Example 5.11 (10 component machine)**.** Suppose you have a machine with 10 independent components in series. The machine only works if all the components work. Each component succeeds with probability $p = 0.95$ and fails with probability $1 - p = 0.05$.

Let Y be the number of components that succeed in a given run of the machine. Then

$$Y \sim \text{Binomial}(n = 10, p = 0.95)$$

Question: what is the probability of the machine working properly?

Example 5.12 (10 component machine, cont'd)**.** What if I arrange these 10 components in parallel? This machine succeeds if at least 1 of the components succeeds.

What is the probability that the new machine succeeds?

Example 5.13 (10 component machine, cont'd)**.** Calculate the expected number of components to succeed and the variance.

Consider a variable

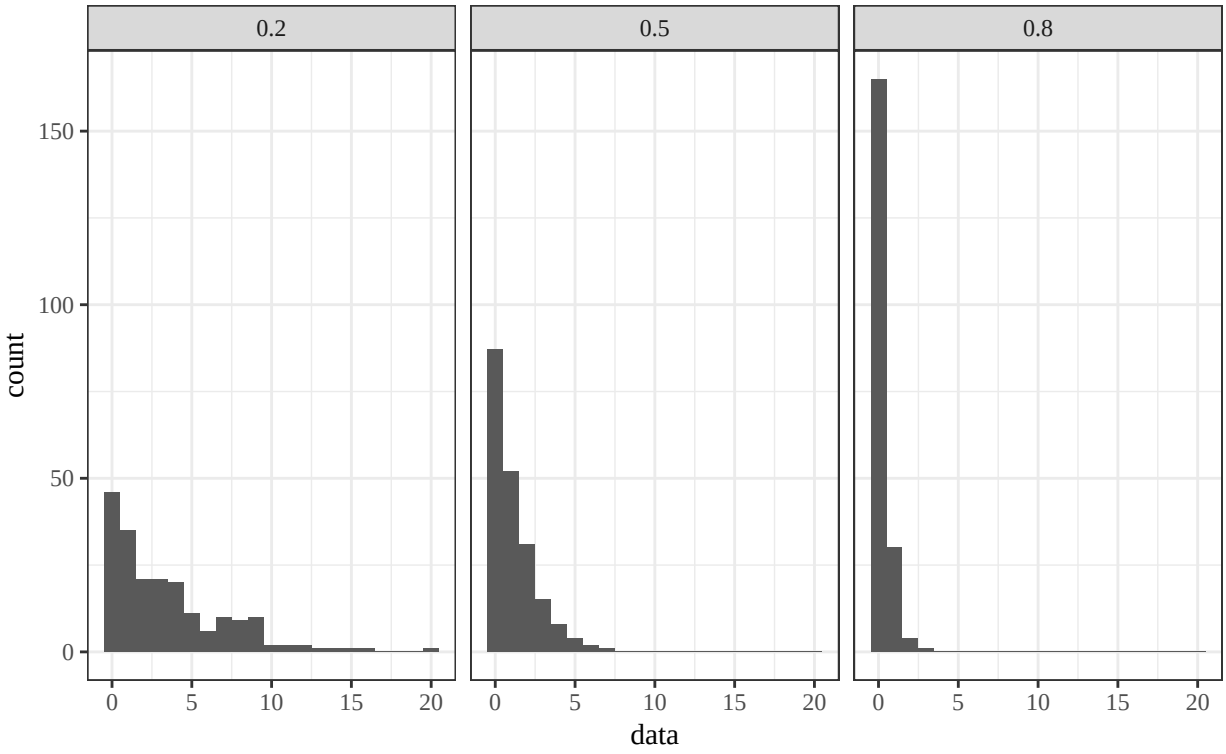
$$X = \text{ the number of trials required to first obtain a success result}$$

Definition 5.10. The *geometric(p) distribution* is a discrete probability distribution with pmf

$$f(x) = \begin{cases} p(1-p)^{x-1} & x = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases}$$

for $0 < p < 1$.

Examples that could follow a binomial(n, p) distribution:



For X a geometric(p) random variable,

$$\mu = \mathrm{E}X = \sum_{x=0}^{n} xp(1-p)^{x-1} = \frac{1}{p}$$
$$\sigma^2 = \mathrm{Var}X = \sum_{x=0}^{n} \left(x - \frac{1}{p}\right)^2 p(1-p)^{x-1} = \frac{1-p}{p^2}$$

Cdf derivation:

Example 5.14 (NiCad batteries)**.** An experimental program was successful in reducing the percentage of manufactured NiCad cells with internal shorts to around 1

Calculate

P(1st or 2nd cell tested has the 1st short)

P(at least 50 cells tested w/o finding a short)

Calculate the expected test number at which the first short is discovered and the variance in test numbers at which the first short is discovered.

It's often important to keep track of the total number of occurrences of some relatively rare phenomenon.

Consider a variable

$X =$ the count of occurences of a phenomenon across a specified interval of time or space

Definition 5.11. The *Poisson(λ) distribution* is a discrete probability distribution with pmf

$$f(x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!} & x = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases}$$

for $\lambda > 0$.

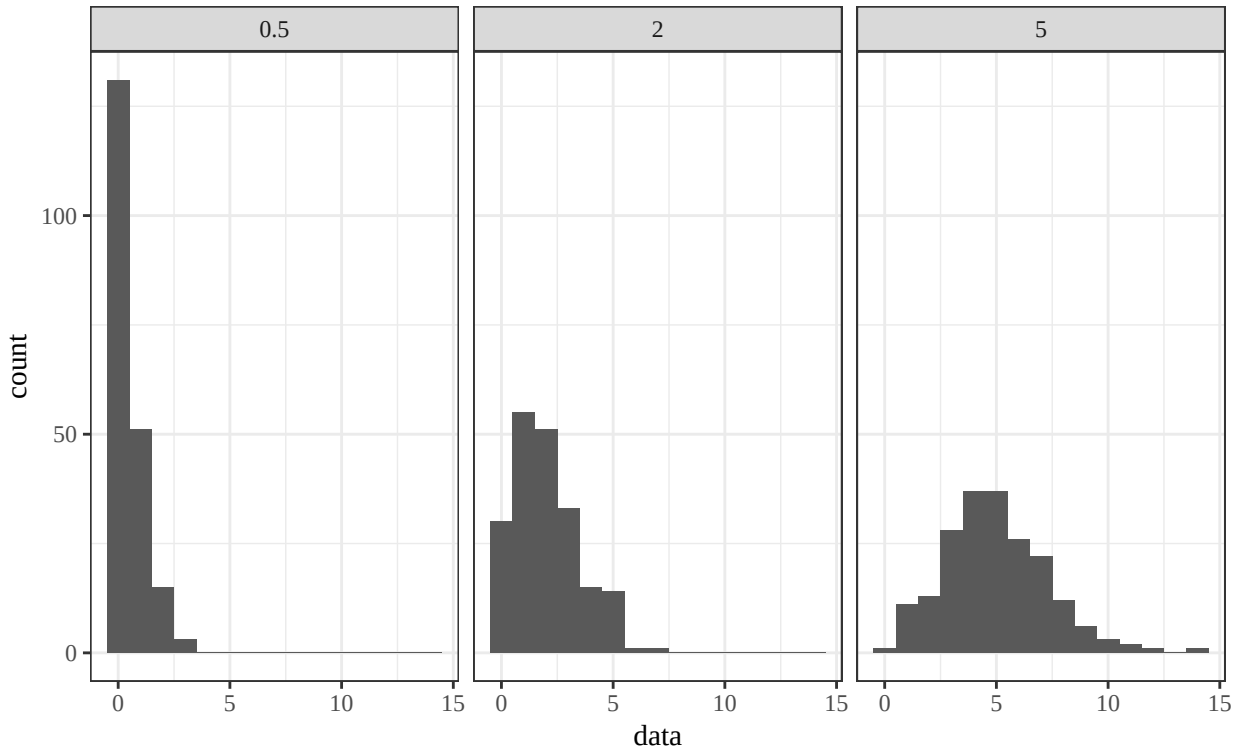
These occurrences must:

1.

2.

3.

Examples that could follow a Poisson(λ) distribution:



For X a Poisson(λ) random variable,

$$\mu = \mathrm{E}X = \sum_{x=0}^{n} x \frac{e^{-\lambda}\lambda^x}{x!} = \lambda$$
$$\sigma^2 = \mathrm{Var}X = \sum_{x=0}^{n} (x-\lambda)^2 \frac{e^{-\lambda}\lambda^x}{x!} = \lambda$$

Example 5.15 (Arrivals at the library)**.** Some students' data indicate that between 12:00 and 12:10pm on Monday through Wednesday, an average of around 125 students entered Parks Library at ISU. Consider modeling

$M =$ the number of students entering the ISU library between 12:00 and 12:01pm next Tuesday

Model $M \sim \text{Poisson}(\lambda)$. What would a reasonable choice of λ be?

Under this model, the probability that between 10 and 15 students arrive at the library between 12:00 and 12:01 PM is:

Example 5.16 (Shark attacks)**.** Let X be the number of unprovoked shark attacks that will occur off the coast of Florida next year. Model $X \sim \text{Poisson}(\lambda)$. From the shark data at http://www.flmnh.ufl.edu/fish/sharks/statistics/FLactivity.htm, 246 unprovoked shark attacks occurred from 2000 to 2009.

What would a reasonable choice of λ be?

Under this model, calculate the following:

$P[\text{no attacks next year}]$

$P[\text{at least 5 attacks}]$

$P[\text{more than 10 attacks}]$