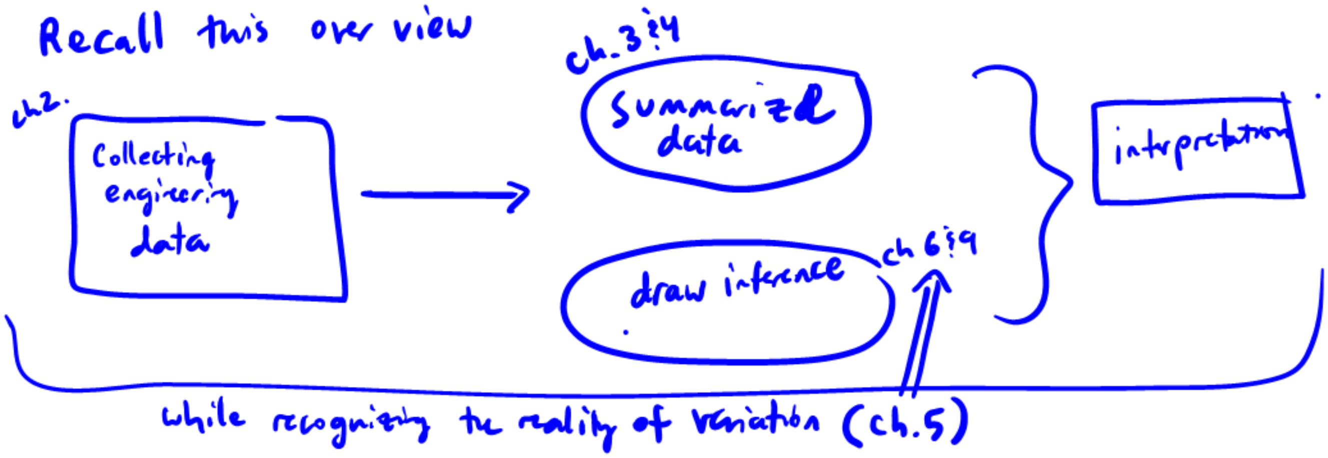
5 Probability: the mathematics of randomness

The theory of probability is the mathematician's description of random variation. This chapter introduces enough probability to serve as a minimum background for making formal statistical inferences.



5.1 (Discrete) random variables

The concept of a random variable is introduced in general terms and the special case of discrete data is considered.

recall:
discrete data - measurements are separated points (# pages in a book)

5.1.1 Random variables and distributions

It is helpful to think of data values as subject to chance influences. Chance is commonly introduced into the data collection process through

1. random sampling techniques
2. measurement error causes
3. changes in system conditions

Definition 5.1. A *random variable* is a quantity that (prior to observation) can be thought of as dependent on chance phenomena.

e.g.
- X = value of a cointoss (heads or tails)
- Z = the amount of torque required to loosen the next bolt
- T = the time you'll have to wait for the next bus home.
- N = the number of defective widgets in a manufacturing process in a day.
- S = the number of unprovoked shark attacks off the coast of Florida next year.

capital letters to stand for r.v.'s.

Definition 5.2. A *discrete random variable* is one that has isolated or separated possible values (rather than a continuum of available outcomes).

Definition 5.3. A *continuous random variable* is one that can be idealized as having an entire (continuous) interval of numbers as its set of values.

Discrete: X, N, S

Continuous: Z, T

Example 5.1 (Roll of a die).

List of possible values

X = roll of a 6-sided fair die – 1, 2, 3, 4, 5, 6

Y = roll of a 6-sided unfair die – 1, 2, 3, 4, 5, 6

How to distinguish between X and Y?

Probability of occurrence!

Definition 5.4. To specify a *probability distribution* for a random variable is to give its set of possible values and (in one way or another) consistently assign numbers between 0 and 1 - called *probabilities* - as measures of the likelihood that the various numerical values will occur

Example 5.2 (Roll of a die, cont'd).

We expect a fair die to land on the number 3 roughly one out of every 6 tosses.

$P[X=3] = \frac{1}{6}$

Suppose the unfair die is weighted so that the number 3 only lands one out of every 22 tosses (on average).

$P[Y=3] = \frac{1}{22}$

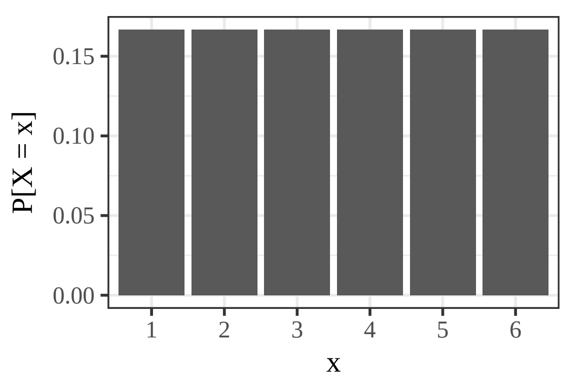
(particular values of RV) (probability that X takes value x)

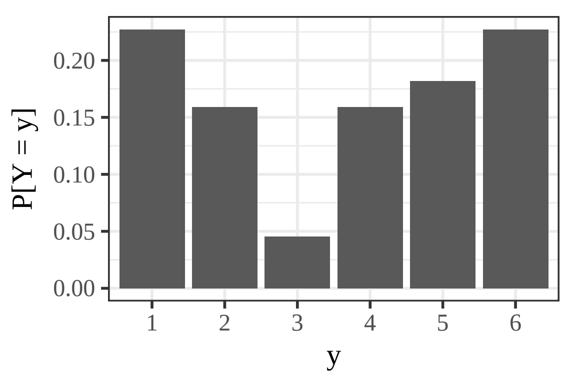
x	1	2	3	4	5	6
$P[X = x]$	1/6	1/6	1/6	1/6	1/6	1/6

fair

y	1	2	3	4	5	6
$P[Y = y]$	5/22	7/44	1/22	7/44	2/11	5/22

unfair



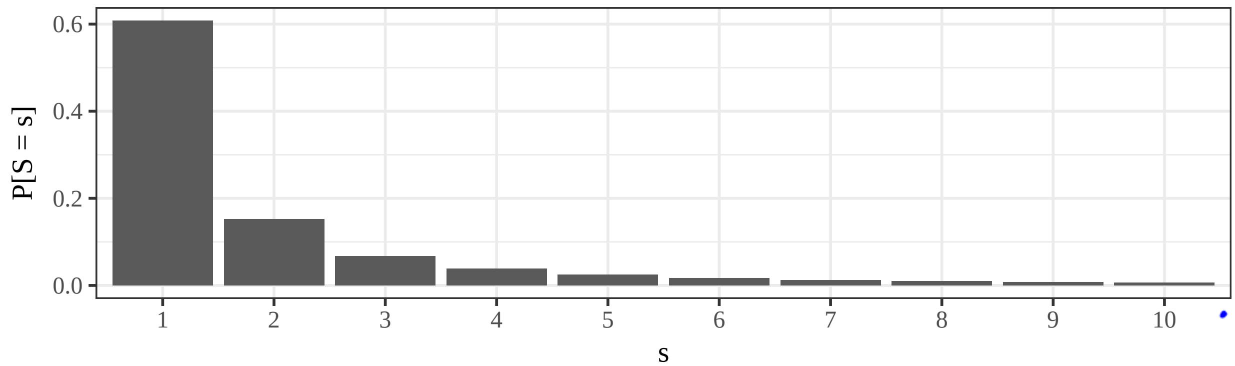


Example 5.3 (Shark attacks)**.** Suppose S is the number of provoked shark attacks off FL next year. This has an infinite number of possible values. Here is one possible (made up) distribution:

S can take values 1, 2, 3, 4, ...

s	1	2	3	$\cdots$	k	$\cdots$
$P[S = s]$	$\frac{6}{\pi^2}$	$\frac{1}{2^2}\frac{6}{\pi^2}$	$\frac{1}{3^2}\frac{6}{\pi^2}$	$\cdots$	$\frac{1}{k^2}\frac{6}{\pi^2}$	$\cdots$

$\sum_{k=1}^{\infty} \frac{1}{k^2}\frac{6}{\pi^2} = 1.$



① All probabilities must be between 0 and 1.

② All probabilities for a random variable must add up to 1 (i.e. the probability of anything happening is 1)

5.1.2 Probability mass functions and cumulative distribution functions

The tool most often used to describe a discrete probability distribution is the *probability mass function*.

Definition 5.5. A *probability mass function (pmf)* for a discrete random variable X, having possible values $x_1, x_2, \ldots$, is a non-negative function $f(x)$ with $f(x_1) = P[X = x_1]$, the probability that X takes the value x_1.

$f(x_i) = P[X = x_i] \quad i = 1, 2, \ldots$

We can also write f_X for the pmf of X
f_S for the pmf of S.

Properties of a mathematically valid probability mass function:

1. $f(x) \geq 0$ for all x (positive probabilities)

2. $\sum_x f(x) = 1$ (probabilities sum to 1).

A probability mass function $f(x)$ gives probabilities of occurrence for individual values. Adding the appropriate values gives probabilities associated with the occurrence of multiple values. $P[X \text{ is either } x_1 \text{ or } x_2] = P[X = x_1] + P[X = x_2] = f(x_1) + f(x_2)$. for $x_1 \neq x_2$.

Example 5.4 (Torque)**.** Let Z = the torque, rounded to the nearest integer, required to loosen the next bolt on an apparatus.

z	11	12	13	14	15	16	17	18	19	20
$f(z)$	0.03	0.03	0.03	0.06	0.26	0.09	0.12	0.20	0.15	0.03

Calculate the following probabilities:

$P(Z \leq 14) = P(Z=11 \text{ or } Z=12 \text{ or } Z=13 \text{ or } Z=14)$
$= P(Z=11) + P(Z=12) + P(Z=13) + P(Z=14)$
$= f(11) + f(12) + f(13) + f(14)$
$= 0.03 + 0.03 + 0.03 + 0.06 = 0.15$

$P(Z > 16) = P(Z=17 \text{ or } Z=18 \text{ or } Z=19 \text{ or } Z=20)$
$= P(Z=17) + P(Z=18) + P(Z=19) + P(Z=20)$
$= f(17) + f(18) + f(19) + f(20)$
$= 0.12 + 0.2 + 0.15 + .03 = 0.5$

$P(Z \text{ is even}) = P(Z=12 \text{ or } Z=14 \text{ or } Z=16 \text{ or } Z=18 \text{ or } Z=20]$
$= f(12) + f(14) + f(16) + f(18) + f(20)$
$= .03 + .06 + .09 + .20 + .03 = .41$

$P(Z \text{ in } \{15, 16, 18\}) = P(Z=15 \text{ or } Z=16 \text{ or } Z=18)$
$= f(15) + f(16) + f(18) = .26 + .09 + .2 = .55$

Another way of specifying a discrete probability distribution is sometimes used.

Definition 5.6. The *cumulative probability distribution (cdf)* for a random variable X is a function $F(x)$ that for each number x gives the probability that X takes that value or a smaller one, $F(x) = P[X \leq x]$.

Since (for discrete distributions) probabilities are calculated by summing values of $f(x)$,

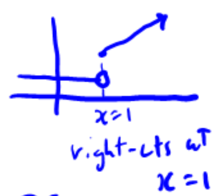
cdf   pmf

$$F(x) = P[X \leq x] = \sum_{y \leq x} f(y)$$

Properties of a mathematically valid cumulative distribution function:

1. $F(x) \geq 0$ for all real numbers x
2. $F()$ is monotonically increasing

3. $F(\ )$ is right continuous



4. $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$

In the discrete case, the graph of $F(x)$ will be a stair-step graph with jumps located at possible values of our random variable and height equal in size to the probabilities associated with those values.

Example 5.5 (Torque, cont'd)**.** Let Z = the torque, rounded to the nearest integer, required to loosen the next bolt on an apparatus.

z	11	12	13	14	15	16	17	18	19	20
$F(z)$	0.03	0.06	0.09	0.15	0.41	0.50	0.62	0.82	0.97	1

.41 − .15 = .26

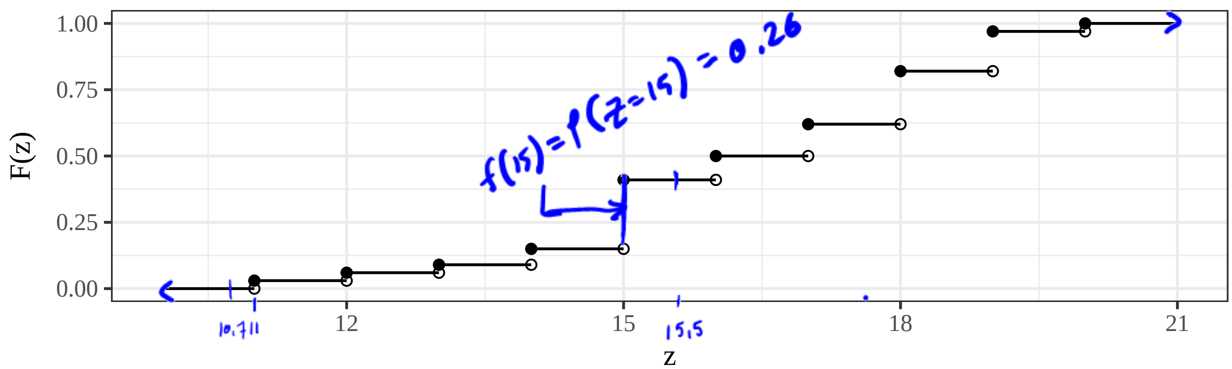


Figure 1: Cdf function for torques.

Calculate the following probabilities using the **cdf only**:

$F(10.7)$ = (definition) $P[Z \le 10.7] = 0$

$P(Z \le 15.5) = P[Z \le 15] = F(15) = 0.41$

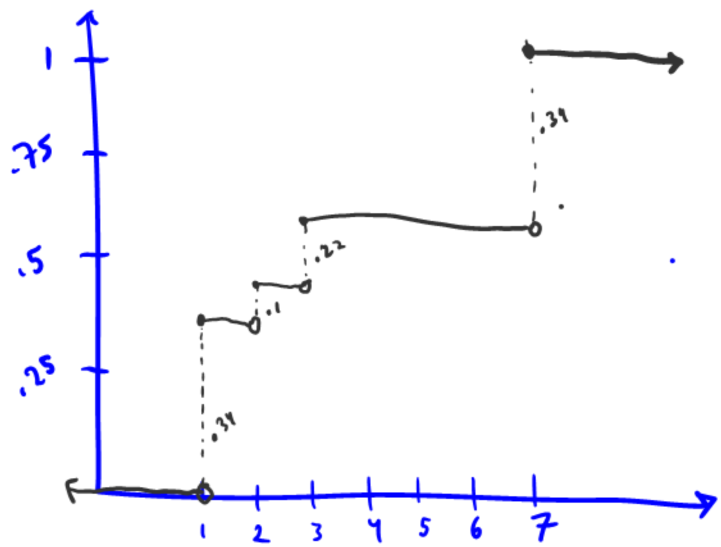
$P(12.1 \le Z \le 14) = P(13 \le Z \le 14) = P(Z=13 \text{ or } Z=14)$
$= f(14) + f(13) = [f(14) + f(13) + f(12) + f(11)] - [f(12) + f(11)]$
$= P[Z \le 14] - P[Z \le 12] = F(14) - F(12) = 0.15 - 0.06 = 0.09$

$P(15 \le Z < 18) = P(Z = 15 \text{ or } 16 \text{ or } 17)$
$= P(Z \le 17) - P(Z \le 14) = F(17) - F(14) = 0.62 - 0.15 = 0.47$

Example 5.6. Say we have a random variable Q with pmf:

q	$f(q)$	$F(q)$
1	0.34	0.34
2	0.1	0.44
3	0.22	0.66
7	0.34	1

Draw the cdf.



5.1.3 Summaries

Almost all of the devices for describing relative frequency (empirical) distributions in Ch. 3 have versions that can describe (theoretical) probability distributions.

1. (measures of location) mean
2. (measures of spread) variance
3. (histograms) probability histograms based on theoretical probabilities

Definition 5.7. The *mean* or *expected value* of a discrete random variable X is

sometimes denoted μ → possible values ↓ probabilities ↙

$$\mathrm{E}X = \sum_x x f(x)$$

EX is the weighted average of the possible values of X, weighted by their probabilities

EX is the mean of the distribution of X.

Example 5.7 (Roll of a die, cont'd)**.** Calculate the expected value of a toss of a fair and unfair die.

x	1	2	3	4	5	6
$P[X = x]$	1/6	1/6	1/6	1/6	1/6	1/6

y	1	2	3	4	5	6
$P[Y = y]$	5/22	7/44	1/22	7/44	2/11	5/22

Fair die:

$$EX = \sum_x x f(x) = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right)$$
$$= 3.5$$

Unfair die:

$$EY = \sum_x x f(x) = 1\left(\frac{5}{22}\right) + 2\left(\frac{7}{44}\right) + 3\left(\frac{1}{22}\right) + 4\left(\frac{7}{44}\right) + 5\left(\frac{2}{11}\right) + 6\left(\frac{5}{22}\right)$$
$$= 3.5909$$

The expected roll of the unfair die is 3.5909.

Example 5.8 (Torque, cont'd)**.** Let Z = the torque, rounded to the nearest integer, required to loosen the next bolt on an apparatus.

z	11	12	13	14	15	16	17	18	19	20
$f(z)$	0.03	0.03	0.03	0.06	0.26	0.09	0.12	0.20	0.15	0.03

Calculate the expected torque required to loosen the next bolt.

$EZ = 11(.03) + 12(.03) + 13(.03) + 14(.06) + 15(.26) + 16(.09) + 17(.12) + 18(.20) + 19(.15) + 20(.03)$

$= 16.35$

The average torque required to loosen the next bolt is 16.35 units.

Definition 5.8. The *variance* of a discrete random variable X is

$EX = \sum x f(x)$

$$\sigma^2 = \mathrm{Var}X = \sum_x (x - \mathrm{E}X)^2 f(x) = \sum_x x^2 f(x) - (\mathrm{E}X)^2.$$

$= E[(X-EX)^2]$ (easy to remember) $\quad E(X^2) - (EX)^2$ (easy to calculate)

[NOTE: if you get a negative value, you have done something wrong]

The *standard deviation* of X is $\sqrt{\mathrm{Var}X}$.

$= SD(X) = \sigma$ (notation)

The variance is the average (in the sense of our distribution) squared deviation of a random variable from its mean.

Example 5.9. Say we have a random variable Q with pmf:

q	$f(q)$
1	0.34
2	0.1
3	0.22
7	0.34

Calculate the variance and the standard deviation.

LONG WAY

$$EQ = 1(.34) + 2(.1) + 3(.22) + 7(.34) = 3.58$$

$$Var\,Q = \sum_q (q - EQ)^2 f(q) = (1-3.58)^2(.34) + (2-3.58)^2(.1) + (3-3.58)^2(.22) + (7-3.58)^2(.34)$$

$$= 6.56$$

SHORT WAY

$$EQ^2 = \sum_q q^2 f(q)$$

$$= 1(.34) + 4(.1) + 9(.22) + 49(.34) = 19.38$$

$$Var\,Q = EQ^2 - (EQ)^2$$

$$= 19.38 - (3.58)^2 = 6.56$$

$$SD(Q) = \sqrt{6.56} = 2.562$$

Example 5.10 (Roll of a die, cont'd)**.** Calculate the variance and standard deviation of a roll of a fair die.

$EX = 3.5$

$E(X^2) = 1\frac{1}{6} + 4\frac{1}{6} + 9\frac{1}{6} + 16\frac{1}{6} + 25\frac{1}{6} + 36\frac{1}{6}$

$= 15.17$

$Var X = EX^2 - (EX)^2$

$= 15.17 - (3.5)^2$

$= 2.92$

$SD(X) = \sqrt{2.92} = 1.7088$

5.1.4 Special discrete distributions

Discrete probability distributions are sometimes developed from past experience with a particular physical phenomenon.

On the other hand, sometimes an easily manipulated set of mathematical assumptions having the potential to describe a variety of real situations can be put together.

One set of assumptions is that of independent identical success-failure trials where

1. There is a constant chance of success outcome on each repetition of the scenario (probability of success p)
2. The repetitions are independent (i.e. knowing the outcome of any one of them does not change assessments of chance related to any others).

Consider a variable

$$X = \text{ the number of successes in } n \text{ independent identical success-failure trials}$$

Definition 5.9. The *binomial*(n, p) *distribution* is a discrete probability distribution with pmf

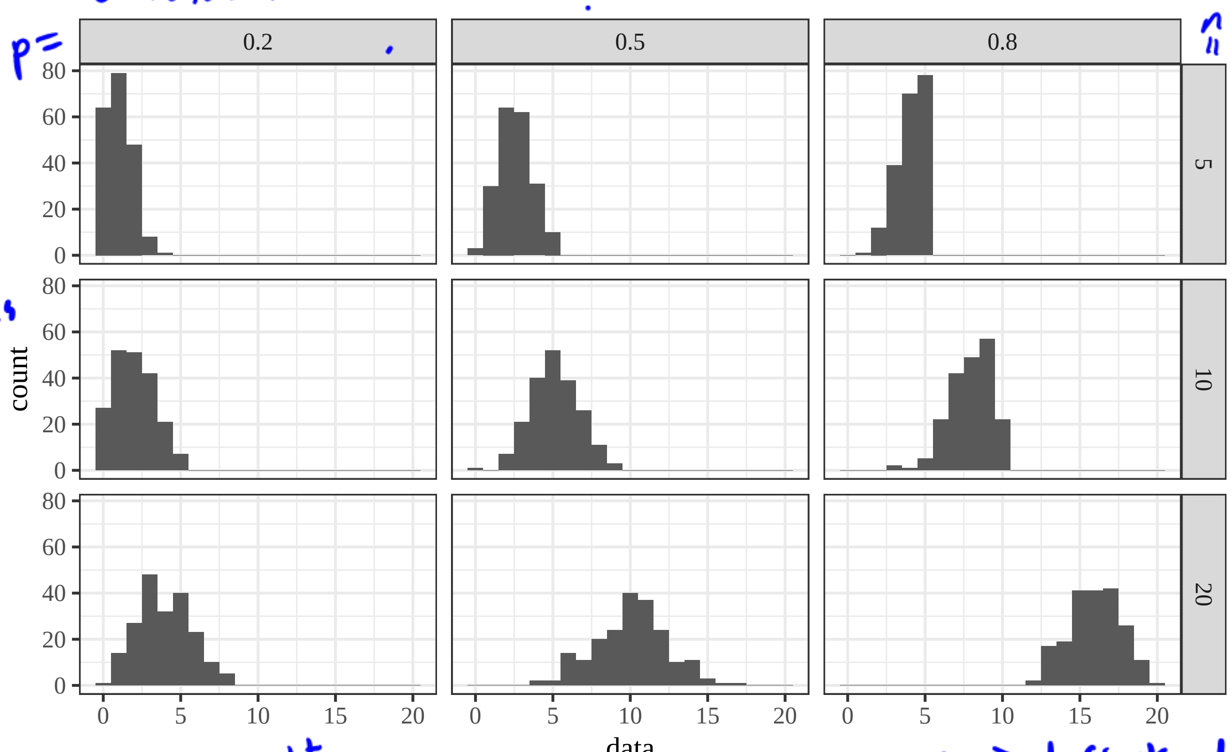
$$f(x) = \begin{cases} \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} & x = 0, 1, \dots, n \\ 0 & \text{otherwise} \end{cases}$$

for n a positive integer and $0 < p < 1$.

(Handwritten notes: "parameters in the distribution" pointing to (n, p); "n choose x": $\binom{n}{x} = \frac{n!}{x!(n-x)!}$, $n! = n(n-1)(n-2)\cdots(3)(2)(1)$; "probability of those values"; "potential values X can take"; "cannot have more than n successes in n trials"; "probability of success" pointing to p.)

Examples that could follow a binomial(n, p) distribution:

- Number of hexamine pellets in a batch of $n=50$ total pellets made from a pelletizing machine that conform to some standard.
- Number of runs of the same chemical process w/ percent yield above 80%, given you run the process $n = 1000$ times.
- Number of rivets that fail in a boiler of $n = 25$ rivets within 3 years of operation. (Note: "success" doesn't have to be good)
- Number of motorists driving over the speed limit at a checkpoint for $n = 1000$ on the road.



For X a binomial(n,p) random variable,

$$\mu = \mathrm{E}X = \sum_{x=0}^{n} x\left(\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}\right) = np$$
$$\sigma^2 = \mathrm{Var}X = \sum_{x=0}^{n}(x-np)^2\left(\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}\right) = np(1-p)$$

$f(x)$ binomial formula

Example 5.11 (10 component machine)**.** Suppose you have a machine with 10 independent components in series. The machine only works if all the components work. Each component succeeds with probability $p = 0.95$ and fails with probability $1 - p = 0.05$.

P(success or failure) = $p + 1 - p = 1$

Let Y be the number of components that succeed in a given run of the machine. Then

$$Y \sim \text{Binomial}(n = 10, p = 0.95)$$

# of trials probability of success

Question: what is the probability of the machine working properly?

P(machine working) = P(all components work)
$= P(Y = 10) = f(10)$
$= \frac{10!}{10!\,0!} .95^{10}(.05)^{10-10}$ $\quad (0! = 1)$
$= .95^{10}$
$= 0.5987$ (not very reliable)

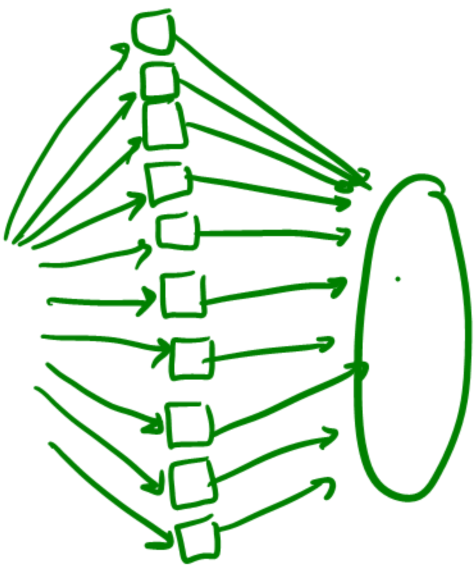
$\frac{n!}{y!(n-y)!} = \frac{10!}{10!(10-10)!}$

$P(\text{succeeding}) + P(\text{failing}) = 1$
(only two options)

Example 5.12 (10 component machine, cont'd)**.** What if I arrange these 10 components in parallel? This machine succeeds if at least 1 of the components succeeds.

What is the probability that the new machine succeeds?

$$
\begin{aligned}
P(\text{new machine succeeds}) &= 1 - P(\text{new machine fails}) \\
&= 1 - P(\text{all components fail}) \\
&= 1 - P(Y=0) = 1 - f(0) \\
&= 1 - \frac{10!}{0!\,10!} .95^0 (.05)^{10-0} \\
&= 1 - .05^{10} \\
&\approx 1
\end{aligned}
$$



Example 5.13 (10 component machine, cont'd)**.** Calculate the expected number of components to succeed and the variance.

$EY = np = 10 \cdot .95 = 9.5$ So the number of components to succeed per run on average is 9.5.

$\text{Var}\,Y = np(1-p) = 10(.95)(.05) = .475$

$SD(Y) = \sqrt{\text{Var}\,Y} = .689$

Consider a variable

$$X = \text{ the number of trials required to first obtain a success result}$$

another generic of r.v.'s

Definition 5.10. The *geometric(p) distribution* is a discrete probability distribution with pmf

1 parameter

1 success; x-1 failures; at least 1 trial

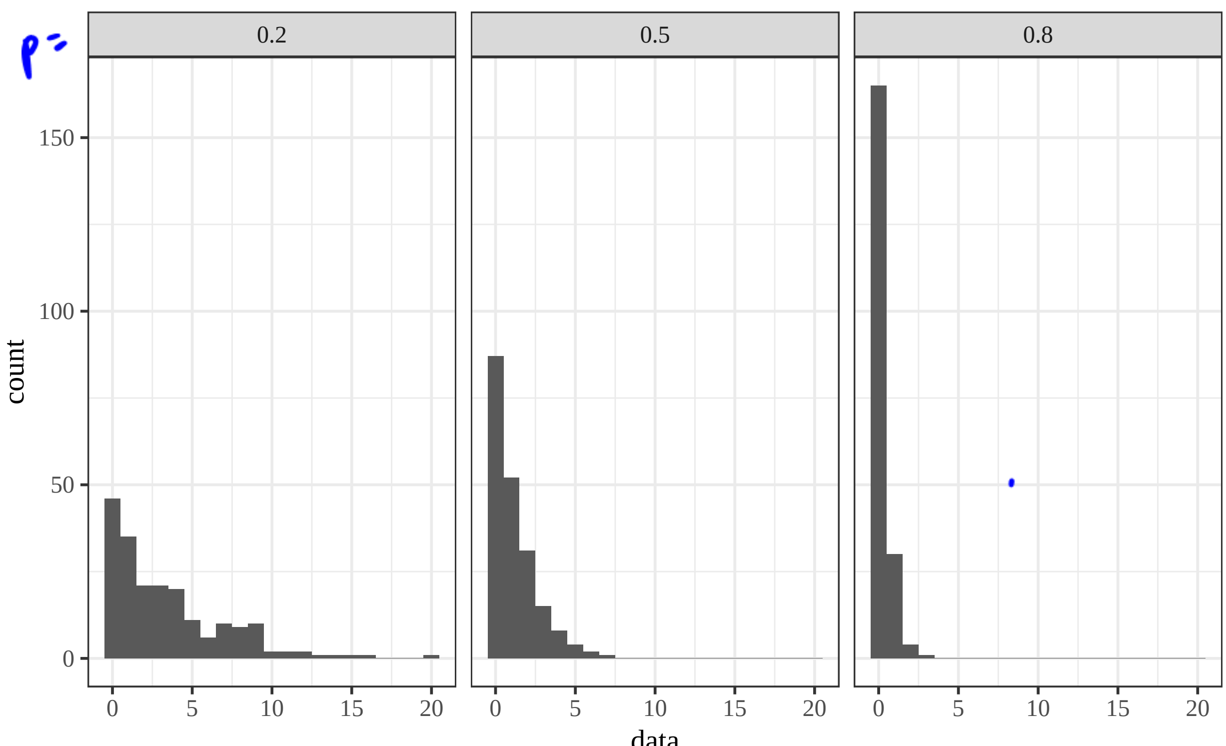
$$f(x) = \begin{cases} p(1-p)^{x-1} & x = 1, \dots \\ 0 & \text{otherwise} \end{cases}$$

for $0 < p < 1$.

Examples that could follow a geometric(p) distribution:

- Number of rolls of a fair die until you land a 5.
- Number of shipments of raw material you get until you get a defective one ("success" does not need to be positive)
- Number of hexamine pellets you make until you make a defective one
- Number of buses that come before yours.

p =



probability decays as x increases (at a faster rate as p increases)

For X a geometric(p) random variable,

geometric series.

$$\mu = \mathrm{E}X = \sum_{x=1}^{\infty} xp(1-p)^{x-1} = \frac{1}{p}$$

$$\sigma^2 = \mathrm{Var}X = \sum_{x=1}^{\infty} \left(x - \frac{1}{p}\right)^2 p(1-p)^{x-1} = \frac{1-p}{p^2}$$

Cdf derivation:

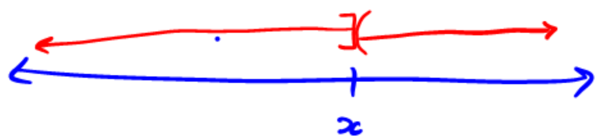
$f(x) = p(1-p)^{x-1}$ for $x = 1, 2, \ldots$ makes intuitive sense. For X to take value x, there must be $x-1$ failures followed by 1 success.

f f f f f f f f | s
x

Another way to think of this:

$F(x) = P(X \le x)$ (by definition)

$1 - F(x) = 1 - P(X \le x)$



$= P(X > x)$

$= P(x$ failure outcomes in x trials$)$ ← Binomial(x, p) r.v.

$= \frac{x!}{0!(x-0)!} p^0 (1-p)^{x-0}$

$= (1-p)^x$

$\Rightarrow 1 - F(x) = (1-p)^x$

$F(x) = 1 - (1-p)^x \quad x = 1, 2, 3, \ldots$

Sometimes useful.

Example 5.14 (NiCad batteries)**.** An experimental program was successful in reducing the percentage of manufactured NiCad cells with internal shorts to around 1%. Let T be the test number at which the first short is discovered. Then, $T \sim \text{Geom}(p)$.

$\uparrow p = .01$

Calculate

$$P(\text{1st or 2nd cell tested has the 1st short}) = P(T=1 \text{ or } T=2)$$
$$= f(1) + f(2)$$
$$= p(1-p)^{1-1} + p(1-p)^{2-1}$$
$$= .01(.99)^0 + .01(.99)^1$$
$$= .02$$

$$P(\text{at least 50 cells tested w/o finding a short}) = P(T > 50) = P(T=51 \text{ or } T=52 \text{ or } T=53 \ldots) = \sum_{x=51}^{\infty} f(x) = \sum_{x=51}^{\infty} p(1-p)^{x-1}$$
$$= 1 - \underbrace{P(T \le 50)}_{\text{CDF}}$$
$$= 1 - F(50)$$
$$= 1 - (1 - (1-p)^{50})$$
$$= (1-p)^{50} = .99^{50}$$
$$= 0.61$$

Calculate the expected test number at which the first short is discovered (ET) and the variance in test numbers at which the first short is discovered ($\text{Var}\,T$).

$ET = \frac{1}{p} = \frac{1}{.01} = 100$ tests for the first short to appear, on average.

$$\text{Var}\,T = \frac{1-p}{p^2} = \frac{.99}{.01^2} = 9900$$

$$SD(T) = \sqrt{\text{Var}\,T} = 99.4987$$

It's often important to keep track of the total number of occurrences of some relatively rare phenomenon.

Consider a variable

$X =$ the count of occurences of a phenomenon across a specified interval of time or space

fixed

1 parameter

Definition 5.11. The *Poisson*(λ) *distribution* is a discrete probability distribution with pmf

$$f(x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!} & x = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases}$$

for $\lambda > 0$.

rate parameter

These occurrences must:

1. be independent
2. be sequential in time (no two occurences at once)
3. occur at the same constant rate, λ

λ, the rate parameter, is the expected number of occurences in the specified interval of time or space. $(EX = \lambda)$

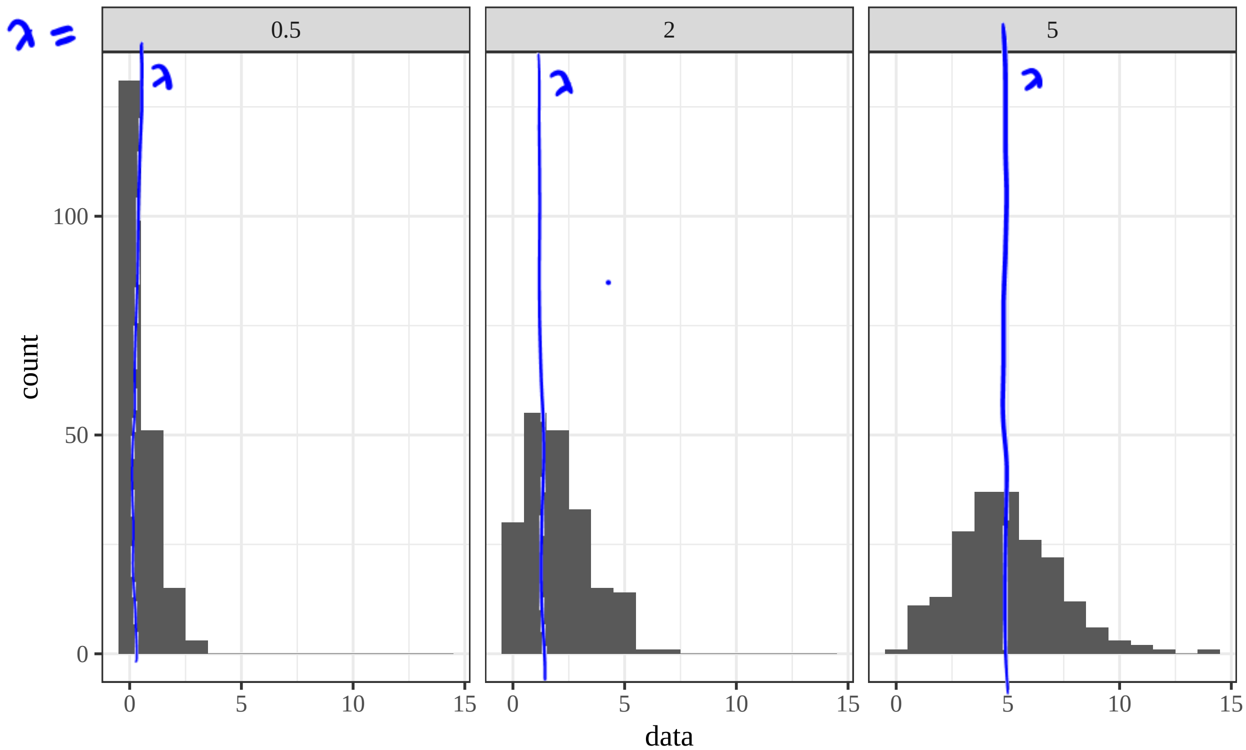
Examples that could follow a Poisson(λ) distribution:

Y is the number of shark attacks off the coast of CA next year, $\lambda = 100$ attacks per year.

Z is the number of shark attacks off the coast of CA next month, $\lambda = 100/12 = 8{,}333$ attacks per month.

N is the number of β particles emitted from a small bar of plutonium, registered by a geiger counter, in a minute, $\lambda = 459.21$ particles per minute.

J is the number of particles per hour, $\lambda = 459.21 \times 60 = 27{,}552.6$ particles per hour.



right skewed with peak near λ

For X a Poisson(λ) random variable,

$$\mu = \mathrm{E}X = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} = \underline{\lambda}$$

$$\sigma^2 = \mathrm{Var}X = \sum_{x=0}^{\infty} (x-\lambda)^2 \frac{e^{-\lambda}\lambda^x}{x!} = \underline{\lambda}$$

Example 5.15 (Arrivals at the library)**.** Some students' data indicate that between 12:00 and 12:10pm on Monday through Wednesday, an average of around 125 students entered Parks Library at ISU. Consider modeling

125 students per 10 minutes

M = the number of students entering the ISU library between 12:00 and 12:01pm next Tuesday

1 minute.

Model $M \sim \text{Poisson}(\lambda)$. What would a reasonable choice of λ be?

$$\lambda = 125/10 = 12.5 \text{ students per minute.}$$

Under this model, the probability that between 10 and 15 students arrive at the library between 12:00 and 12:01 PM is:

$$P(10 \le M \le 15) = f(10) + f(11) + f(12) + f(13) + f(14) + f(15)$$

$$= \frac{e^{-12.5}(12.5)^{10}}{10!} + \frac{e^{-12.5}(12.5)^{11}}{11!} + \dots + \frac{e^{-12.5}(12.5)^{15}}{15!}$$

$$= 0.6$$

Example 5.16 (Shark attacks). Let X be the number of unprovoked shark attacks that will occur off the coast of Florida next year. Model $X \sim \text{Poisson}(\lambda)$. From the shark data at http://www.flmnh.ufl.edu/fish/sharks/statistics/FLactivity.htm, 246 unprovoked shark attacks occurred from 2000 to 2009.

(handwritten: "fixed interval" under "next year"; "246 attacks in 10 years")

What would a reasonable choice of λ be?

$$\lambda = 246/10 = 24.6 \text{ attacks per year}$$

Under this model, calculate the following:

(handwritten: P(I go to FL next year) =)

P[no attacks next year] $= P(X=0) = f(0) = e^{-24.6}\frac{(24.6)^0}{0!} \approx 2.07 \times 10^{-11}$

P[at least 5 attacks] $= P(X \geq 5) = 1 - P(X<5) = 1 - [f(0)+f(1)+f(2)+f(3)+f(4)]$

$$= 1 - e^{-24.6}\left(\frac{24.6^0}{0!} + \frac{24.6^1}{1!} + \frac{24.6^2}{2!} + \frac{24.6^3}{3!} + \frac{24.6^4}{4!}\right) \approx 0.999996$$

(handwritten hint: $\sum_{i=0}^{10} \frac{24.6^i}{i!} = 36266812$)

P[more than 10 attacks] $= P(X>10) = 1 - P(X \leq 10) = 1 - \sum_{x=0}^{10} f(x)$

$$= 1 - e^{-24.6}\sum_{x=0}^{10}\frac{24.6^x}{x!} = 1 - e^{-24.6}(36266812) = 0.999249$$

5.2 Continuous random variables

It is often convenient to think of a random variable as having a whole (continuous) interval for its set of possible values.

The devices used to describe continuous probability distributions differ from those that describe discrete probability distributions.

Examples of continuous random variables:

Z = the amount of torque required to loosen the next bolt (not rounded)

T = time you'll wait for the next bus

C = the outside temperate at 11:39 AM tomorrow

L = the length of the next manufactured part

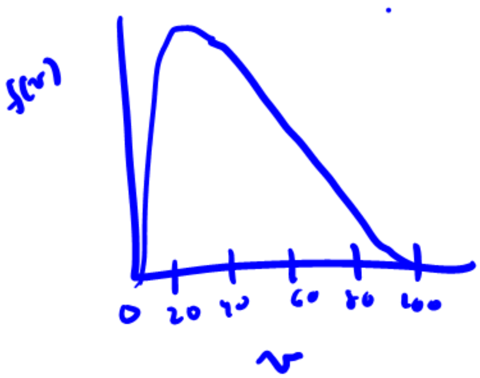
V = % yield of the next run of process

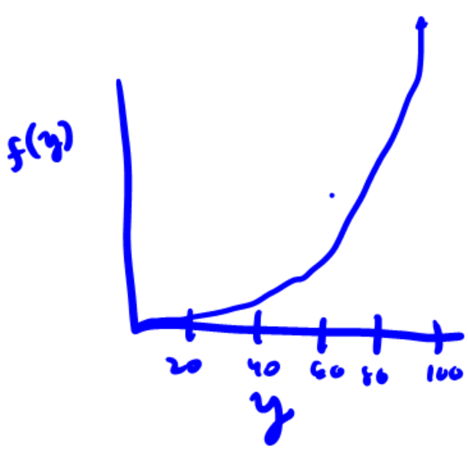
Y = % yield of the next run of a different

↳ how do we mathematically distinguish between V and Y?

- Each has the same range $0\% \le V, Y \le 100\%$
- There are uncountably many possible values in this range. (can't make a table of probabilities)

Distribution!





The process Y will yield more product per run on average than process V

5.2.1 Probability density functions and cumulative distribution functions

A *probability density function (pdf)* is the continuous analogue of a discrete random variable's probability mass function (pmf).

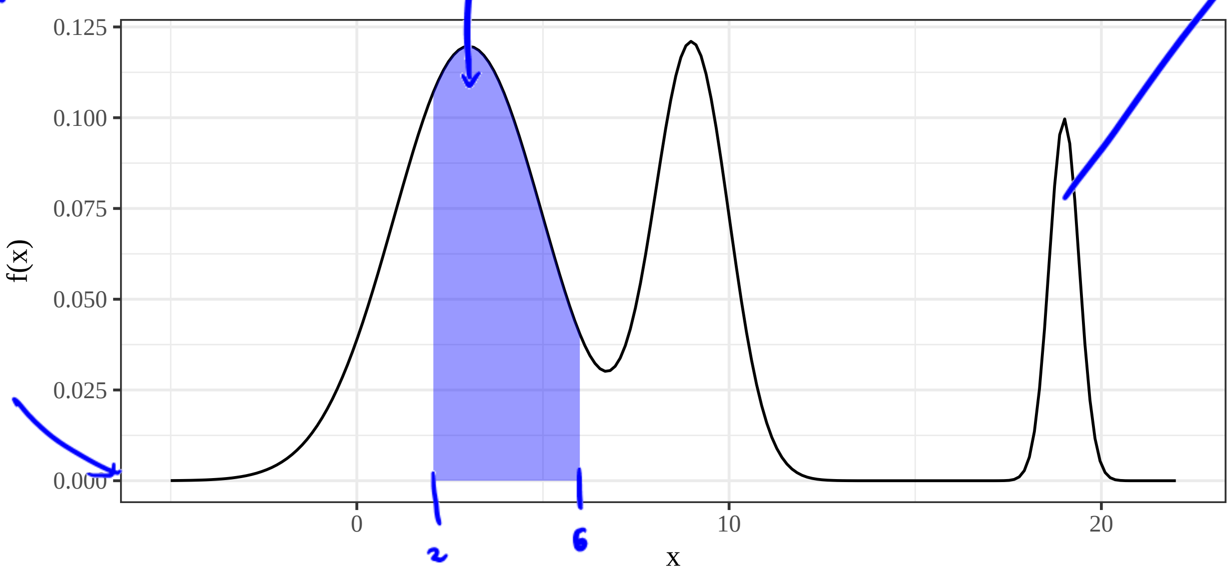
Definition 5.12. A *probability density function (pdf)* for a continuous random variable X is a nonnegative function $f(x)$ with

$$\int_{-\infty}^{\infty} f(x) = 1$$

and such that for all $a \leq b$,

$$P[a \leq X \leq b] = \int_{a}^{b} f(x)dx.$$

1. $f(x) \geq 0 \quad \forall x$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$
3. $P[a \leq X \leq b] = \int_a^b f(x)dx, \quad a \leq b$



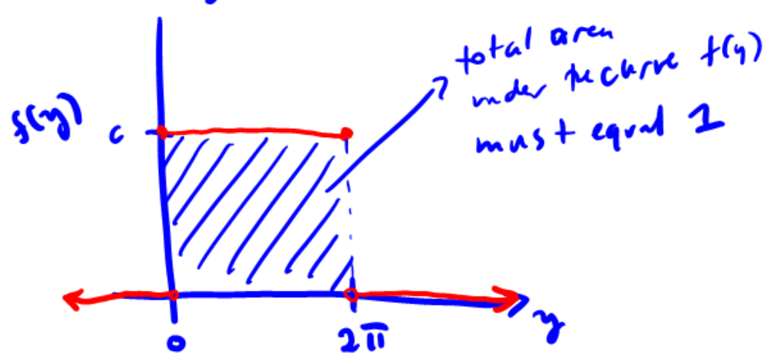
Example 5.17 (Compass needle)**.** Consider a de-magnetized compass needle mounted at its center so that it can spin freely. It is spun clockwise and when it comes to rest the angle, θ, from the vertical, is measured. Let

$$Y = \text{the angle measured after each spin in radians}$$

What values can Y take? $[0, 2\pi]$

What form makes sense for $f(y)$? $f(y) = \begin{cases} c & 0 \le y \le 2\pi \\ 0 & \text{otherwise} \end{cases}$

Y has a positive probability between 0 and 2π and it is equally likely to land at any angle (can spin freely)



We say that Y is distributed "Uniform$(0, 2\pi)$"

If this form is adopted, that what must the pdf be?

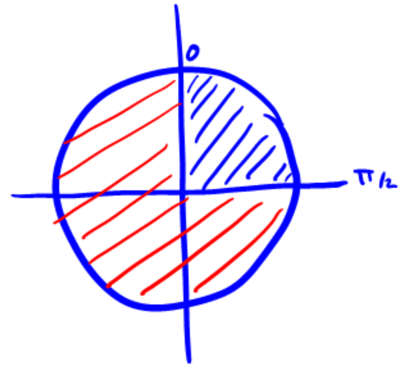
$$1 = \int_{-\infty}^{\infty} f(y)\,dy = \int_{-\infty}^{0} 0\,dy + \int_{0}^{2\pi} c\,dy + \int_{2\pi}^{\infty} 0\,dy$$

$$= cy\Big|_0^{2\pi} = 2\pi c$$

$$\Rightarrow c = \frac{1}{2\pi}$$

This $f(y) = \begin{cases} \frac{1}{2\pi} & 0 \le y \le 2\pi \\ 0 & \text{otherwise} \end{cases}$

Using this pdf, calculate the following probabilities:



1. $P[Y < \frac{\pi}{2}] = P(-\infty < Y < \frac{\pi}{2})$

$$= \int_{-\infty}^{\pi/2} f(y)\,dy$$
$$= \int_{-\infty}^{0} 0\,dy + \int_0^{\pi/2} \frac{1}{2\pi}\,dy$$
$$= \frac{1}{2\pi} y \Big|_0^{\pi/2} = \frac{1}{2\pi}\cdot\frac{\pi}{2} = \frac{1}{4} = 0.25$$

2. $P[\frac{\pi}{2} < Y < 2\pi] = \int_{\pi/2}^{2\pi} f(y)\,dy$

$1 - P[Y < \frac{\pi}{2}]$
$= 1 - \frac{1}{4} = \frac{3}{4}$

$$= \int_{\pi/2}^{2\pi} \frac{1}{2\pi}\,dy$$
$$= \frac{1}{2\pi}\cdot 2\pi - \frac{1}{2\pi}\cdot \pi/2 = 1 - \frac{1}{4} = \frac{3}{4}$$

3. $P[\frac{\pi}{6} < Y < \frac{\pi}{4}] = \int_{\pi/6}^{\pi/4} f(y)\,dy$

$$= \int_{\pi/6}^{\pi/4} \frac{1}{2\pi}\,dy$$
$$= \frac{1}{2\pi}\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{1}{2\pi}\cdot\frac{1}{12} = \frac{1}{24} \approx .04167$$

4. $P[Y = \frac{\pi}{6}] = P[\frac{\pi}{6} \le Y \le \frac{\pi}{6}]$

$$= \int_{\pi/6}^{\pi/6} f(y)\,dy$$
$$= \int_{\pi/6}^{\pi/6} \frac{1}{2\pi}\,dy$$
$$= \frac{1}{2\pi}\left(\frac{\pi}{6} - \frac{\pi}{6}\right) = 0$$

In fact, for any continuous random variable X and any real number a,

$$P[X = a] = 0.$$

Definition 5.13. The *cumulative distribution function (cdf)* of a continuous random variable X is a function F such that

$$F(x) = P[X \leq x] = \int_{-\infty}^{x} f(t)dt$$

same for discrete or cts

$F(x)$ is obtained from $f(x)$ by integration, and applying the fundamental theorem of calculus yields

$$\frac{d}{dx}F(x) = f(x).$$

That is, $f(x)$ is obtained from $F(x)$ by differentiation.

As with discrete random variables, F has the following properties:

1. $F(x) \geq 0$ for all real x

2. F is monotonically increasing (i.e. is never decreasing)

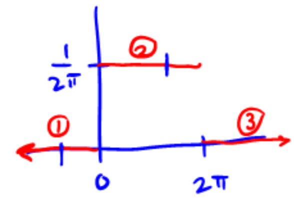
3. $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$

Different

4. F is continuous. (instead of just right continuous)

Example 5.18 (Compass needle, cont'd)**.** Recall the compass needle example, with

$$f(y) = \begin{cases} \frac{1}{2\pi} & 0 \le y \le 2\pi \\ 0 & \text{otherwise} \end{cases}$$



Find the cdf.

For $y < 0$

$$F(y) = P[Y \le y] = \int_{-\infty}^{y} f(y)\,dy = \int_{-\infty}^{y} 0\,dy = 0$$

For $0 \le y \le 2\pi$

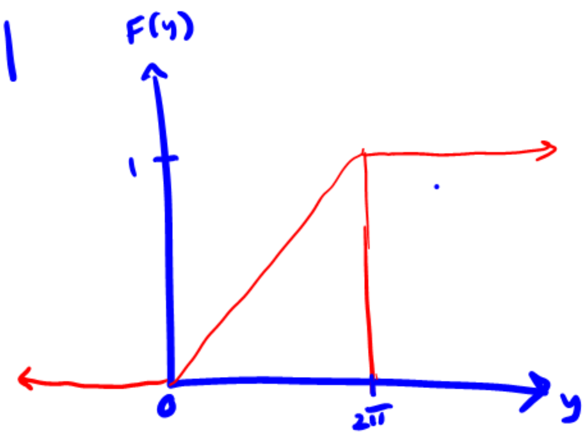
$$F(y) = P(Y \le y] = \int_{-\infty}^{y} f(y)\,dy = \int_{-\infty}^{0} 0\,dy + \int_{0}^{y} \frac{1}{2\pi}\,dy = \frac{y}{2\pi}$$

For $y > 2\pi$

$$F(y) = P(Y \le y) = \int_{-\infty}^{y} f(y)\,dy = \int_{-\infty}^{\infty} 0\,dy + \int_{0}^{2\pi} \frac{1}{2\pi}\,dy + \int_{2\pi}^{y} 0\,dy$$

$$= 1$$

$$F(y) = \begin{cases} 0 & y < 0 \\ y/2\pi & 0 \le y \le 2\pi \\ 1 & y > 2\pi \end{cases}$$



Calculate the following using the cdf:

$F(1.5)$

$0 \le 1.5 \le 2\pi \Rightarrow F(1.5) = \frac{1.5}{2\pi} = \frac{3}{4\pi} \approx 0.2387$

$P[Y \le \frac{4\pi}{5}] = F(\frac{4\pi}{5}) = \frac{4\pi}{5} \cdot \frac{1}{2\pi} = \frac{2}{5} = 0.4$

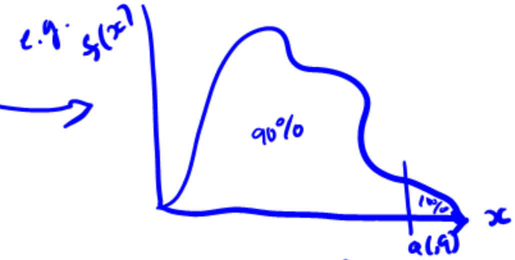
Aside:

$1 = P(\text{everything}) = P(X > x \text{ or } X \le x)$
$= P(X > x) + P(X \le x)$

For any random variable X, discrete or cts, $P(X > x) = 1 - P(X \le x)$

$P[Y > 5] = 1 - P[Y \le 5]$
$= 1 - F(5) = 1 - \frac{5}{2\pi} \approx 0.2042$

$P[\frac{\pi}{3} < Y \le \frac{\pi}{2}] = \int_{\pi/3}^{\pi/2} f(y)dy = \int_{-\infty}^{\pi/2} f(y)dy - \int_{-\infty}^{\pi/3} f(y)dy$
$= F(\pi/2) - F(\frac{\pi}{3})$
$= \frac{\pi}{2} \cdot \frac{1}{2\pi} - \frac{\pi}{3} \cdot \frac{1}{2\pi}$
$= \frac{1}{4} - \frac{1}{6} = \frac{1}{12} \approx 0.08333$



5.2.2 Quantiles

Recall: The p quantile of a distribution of data is a number such that a fraction p of the distribution lies to the left of that number and a fraction 1−p of the distribution lies to the right.

Definition 5.14. The *p-quantile of a random variable*, X, is the number $Q(p)$ such that $P[X \leq Q(p)] = p$.

In terms of the cumulative distribution function (for a continuous random variable),

$$p = P[X \leq Q(p)] = F(Q(p))$$

cdf is cts and monotonically increasing ⟹ inverses

i.e. $F^{-1}(p) = Q(p)$

inverse function, not $\frac{1}{F(p)}$

Example 5.19 (Compass needle, cont'd)**.** Recall the compass needle example, with

$$f(y) = \begin{cases} \frac{1}{2\pi} & 0 \leq y \leq 2\pi \\ 0 & \text{otherwise} \end{cases}$$

$Q(.95)$:

$$0.95 = P(Y \leq Q(.95))$$
$$= F(Q(.95))$$
$$= \frac{Q(.95)}{2\pi}$$

solve for $Q(.95)$ is where we are finding F^{-1}

$$\Rightarrow Q(.95) = .95(2\pi) \approx 5.969$$

On average, 95% of needle spins will land at 5.969 radians or below.

You can also calculate quantiles directly from the cdf.

$$F(y) = \begin{cases} 0 & y < 0 \\ \frac{1}{2\pi} y & 0 \leq y \leq 2\pi \\ 1 & \text{otherwise} \end{cases}$$

$Q(.25)$:

$$0.25 = P[Y \leq Q(.25)] = F(Q(.25)) = \frac{Q(.25)}{2\pi}$$

$$\Rightarrow Q(.25) = .25(2\pi) = \frac{\pi}{2} \approx 1.5708 \text{ radians}$$

"median" → $Q(.5)$

def'n of quantiles; def'n of cdf; cdf formula

$$0.5 = P[Y \leq Q(.5)] = F(Q(.5)) = \frac{Q(.5)}{2\pi}$$

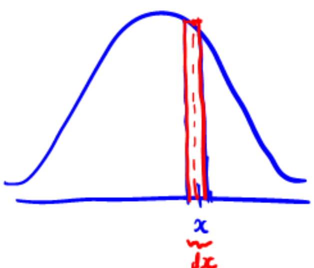
$$\Rightarrow Q(.5) = 0.5(2\pi) = \pi \approx 3.1416 \text{ radians.}$$

5.2.3 Means and variances for continuous distributions

It is possible to summarize continuous probability distributions using

1. plot of probability density function (pdf) $f(x)$ (kind of an idealized histogram where bin sizes are → 0)
2. measure of location (mean or expected value)
3. measure of spread (variance or standard deviation)

Definition 5.15. The *mean* or *expected value* of a continuous random variable X is



$$\mu = \text{E}X = \int_{-\infty}^{\infty} x f(x) dx.$$

sometimes denoted μ

The probability in a small interval around x of length dx as $f(x)dx$

$\text{E}X \approx \sum x f(x) dx$ and

$\text{E}X = \lim_{dx \to 0} \sum x f(x) dx = \int x f(x) dx$

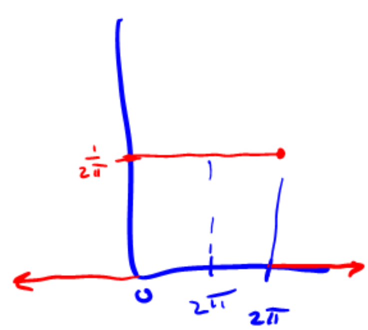
Example 5.20 (Compass needle, cont'd)**.** Calculate EY where Y is the angle from vertical in radians that a spun needle lands on.

$$f(y) = \begin{cases} \frac{1}{2\pi} & 0 \le y \le 2\pi \\ 0 & \text{otherwise} \end{cases}$$

$\text{E}Y = \int_{-\infty}^{\infty} y f(y) dy$

$= \int_{-\infty}^{0} y \cdot 0 \, dy + \int_{0}^{2\pi} y \cdot \frac{1}{2\pi} dy + \int_{2\pi}^{\infty} y \cdot 0 \, dy$

$= \left[\frac{y^2}{4\pi} \right]_0^{2\pi} = \frac{(2\pi)^2}{4\pi} = \pi.$



Notice $\text{E}Y = Q(.5)$, uniform distribution is symmetric.

Example 5.21. Calculate EX where X follows the following distribution

$$f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{3}e^{-x/3} & x \geq 0 \end{cases}$$

$$EX = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} \frac{1}{3} x e^{-x/3}\,dx$$

$u = x \qquad dv = \frac{1}{3}e^{-x/3}\,dx$

$u' = 1 \qquad v = -e^{-x/3}$

$$= -xe^{-x/3}\Big|_0^{\infty} - \int_0^{\infty} -e^{-x/3}\,dx$$

$$= -xe^{-x/3}\Big|_0^{\infty} + -3e^{-x/3}\Big|_0^{\infty}$$

$$= \lim_{a\to\infty} -ae^{-a/3} + 0\cdot e^{-0/3} - 3\cdot 0 + 3e^0 = 3$$

because x goes to ∞ slower than e^{-x} goes to 0. (L'Hopital's rule)

$+\ x$	$\frac{1}{3}e^{-x/3}$
$-\ 1$	$-e^{-x/3}$
$+\ 0$	$3e^{-x/3}$

$$= -xe^{-x/3} - 3e^{-x/3}\Big|_0^{\infty}$$

$$= 3$$

Definition 5.16. The *variance* of a continuous random variable X is

$$\text{Var}X = E[(X-EX)^2] = \int_{-\infty}^{\infty} (x - \text{E}X)^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - (\text{E}X)^2. = EX^2 - (EX)^2$$

easy to compute

The *standard deviation* of X is $\sqrt{\text{Var}X}$. SD(X)

Example 5.22 (Library books)**.** Let X denote the amount of time for which a book on 2-hour hold reserve at a college library is checked out by a randomly selected student and suppose its density function is

$$f(x) = \begin{cases} 0.5x & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Calculate EX and Var$X$.

$$EX = \int_{-\infty}^{\infty} x f(x) dx = \int_0^2 x \cdot 0.5x \, dx$$

$$= \int_0^2 0.5 x^2 dx$$

$$= \left[\frac{0.5}{3} x^3 \right]_0^2 = \frac{8}{6} \approx 1.3333$$

$$EX^2 = \int_{-\infty}^{\infty} x^2 f(x) dx$$

$$= \int_0^2 0.5 x^3 dx$$

$$= \left[\frac{0.5}{4} x^4 \right]_0^2 = \frac{16}{8} = 2$$

$$\text{Var} X = EX^2 - (EX)^2$$

$$= 2 - \left(\frac{8}{6}\right)^2$$

$$= \frac{2}{9}$$

$$SD(X) = \frac{\sqrt{2}}{3}$$

Example 5.23 (Ecology)**.** An ecologist wishes to mark off a circular sampling region having radius 10m. However, the radius of the resulting region is actually a random variable R with pdf

$$f(r) = \begin{cases} \frac{3}{2}(10-r)^2 & 9 \le r \le 11 \\ 0 & \text{otherwise} \end{cases}$$

Calculate ER and SD(R).

$$ER = \int_{-\infty}^{\infty} r f(r)\,dr$$
$$= \int_9^{11} r \cdot \frac{3}{2}(10-r)^2\,dr$$
$$= \frac{3}{2}\int_9^{11}(100r - 20r^2 + r^3)\,dr$$
$$= \frac{3}{2}\left[100\frac{r^2}{2} - 20\frac{r^3}{3} + \frac{r^4}{4}\right]_9^{11}$$
$$= \frac{3}{2}\left[100\cdot\frac{(11)^2}{2} - 20\frac{11^3}{3} + \frac{11^4}{4} - 100\cdot\frac{9^2}{2} + 20\frac{9^3}{3} - \frac{9^4}{4}\right]$$
$$= 10$$

$$ER^2 = \int_{-\infty}^{\infty} r^2 f(r)\,dr$$
$$= \int_9^{11} r^2 \cdot \frac{3}{2}(10-r)^2\,dr$$
$$= \frac{3}{2}\int_9^{11}(100r^2 - 20r^3 + r^4)\,dr$$
$$= \frac{3}{2}\left[100\frac{r^3}{3} - 20\frac{r^4}{4} + \frac{r^5}{5}\right]_9^{11}$$
$$= 100.6$$

$$\text{Var}R = ER^2 - (ER)^2 = 100.6 - 10^2 = 0.6$$
$$SD(R) = \sqrt{\text{var}R} = \sqrt{.6} \approx 0.7746$$

Why does $\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$?

For any function g of a cts r.v. X, with pdf $f(x)$

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

So, $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$ where $g(X) = X^2$

Also,
For any function g of a discrete r.v. X, with pmf $f(x)$

$$E(g(X)) = \sum_x g(x) f(x)$$

Example 5.24 (Ecology, cont'd)**.** Calculate the expected *area* of the circular sampling region.

R = the radius of the circular sampling region
A = the area of the circular sampling region = $g(R) = \pi R^2$

$$EA = Eg(R) = E\pi R^2 = \int_{-\infty}^{\infty} \pi r^2 f(r)\, dr$$

$$= \pi \int_{-\infty}^{\infty} r^2 f(r)\, dr \quad = ER^2 \text{ (previous page)}$$

$$= \pi \cdot 100.6$$

For a linear function, $g(X) = aX + b$, where a and b are constants,

$$\mathrm{E}(aX + b) = \int_{-\infty}^{\infty} (ax+b) f(x)\,dx$$

$$= \int_{-\infty}^{\infty} ax f(x)\,dx + \int_{-\infty}^{\infty} b f(x)\,dx$$

$$= a\underbrace{\int_{-\infty}^{\infty} x f(x)\,dx}_{EX} + b\underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{1}$$

$$= aEX + b$$

$$\mathrm{Var}(aX + b) = E\left((aX+b)^2\right) - \left(E(aX+b)\right)^2$$

$$= E\left(a^2X^2 + 2abX + b^2\right) - \left(aEX + b\right)^2$$

$$= Ea^2X^2 + E2abX + Eb^2 - \left(a^2(EX)^2 + 2abEX + b^2\right)$$

$$= a^2EX^2 + 2abEX + b^2 - a^2(EX)^2 - 2abEX - b^2$$

$$= a^2EX^2 - a^2(EX)^2$$

$$= a^2\left(EX^2 - (EX)^2\right)$$

$$= a^2 \mathrm{Var}X$$

Example 5.25 (Ecology, cont'd)**.** Calculate the expected value and variance of the *diameter* of the circular sampling region.

R = the radius of the circular sampling region

D = diameter = $g(R) = \underbrace{2}_{a} \cdot R + \underbrace{0}_{b}$

$$E(D) = E(g(R)) = E(2R + 0) = 2ER + 0 = 20$$

$$\mathrm{Var}(D) = \mathrm{Var}(g(R)) = \mathrm{Var}(2R + 0) = 2^2\,\mathrm{Var}R = 4(.6) = 2.4$$

Definition 5.17. *Standardization* is the process of transforming a random variable, X, into the signed number of standard deviations by which it is is above its mean value.

$$Z = \frac{X - \mathrm{E}X}{\mathrm{SD}(X)}$$

← by subtracting the mean and dividing by the standard deviation

Z has mean 0

$$EZ = E\left(\frac{X - EX}{SD(X)}\right) = E\left(\frac{1}{SD(X)}X - \frac{EX}{SD(X)}\right)$$

$a = \frac{1}{SD(X)}$

$b = -\frac{EX}{SD(X)}$

$$= \frac{1}{SD(X)} E(X) - \frac{EX}{SD(X)}$$

$$= 0$$

Z has variance (and standard deviation) 1

$$\mathrm{Var}\, Z = \mathrm{Var}\left(\frac{X - EX}{SD(X)}\right)$$

$$= \mathrm{Var}\left(\frac{1}{SD(X)}X - \frac{EX}{SD(X)}\right)$$

$a = \frac{1}{SD(X)}$ $\quad b = -\frac{EX}{SD(X)}$

$$= \left(\frac{1}{SD(X)}\right)^2 \mathrm{Var} X$$

$$= \frac{1}{\mathrm{Var} X} \mathrm{Var} X = 1$$

5.2.4 A special continuous distribution

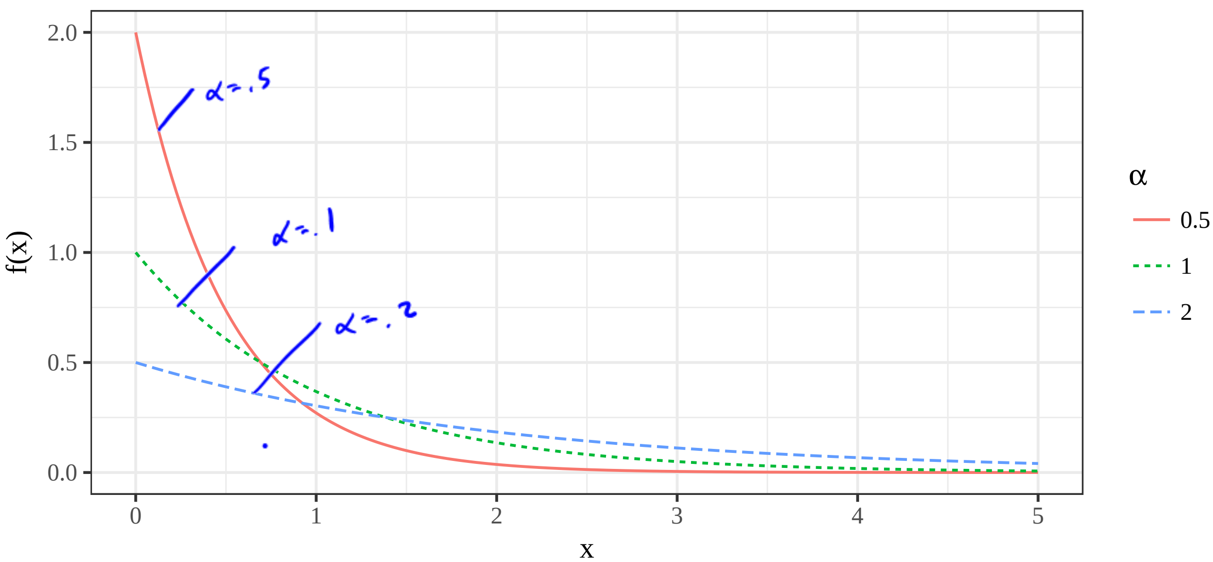
Just as there are a number of useful discrete distributions commonly applied to engineering problems, there are a number of standard continuous probability distributions.

Definition 5.18. The *exponential*(α) *distribution* is a continuous probability distribution with probability density function

Example 5.21 is an Exp(3) dsn

$$f(x) = \begin{cases} \frac{1}{\alpha} e^{-x/\alpha} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

for $\alpha > 0$.



An Exp(α) random variable measures the waiting time until a specific event that has an equal chance of happening at any point in time. This is kind of like a continuous geometric distribution

Examples:

- Time between your arrival at the bus stop and the moment the bus comes.
- Time until the next person walks inside the library.
- Time until the next car accident on a stretch of highway.

It is straightforward to show for $X \sim \text{Exp}(\alpha)$,

1. $\mu = \text{E}X = \int_0^\infty x \frac{1}{\alpha} e^{-x/\alpha} dx = \alpha$

2. $\sigma^2 = \text{Var}X = \int_0^\infty (x-\alpha)^2 \frac{1}{\alpha} e^{-x/\alpha} dx = \alpha^2$

Further, $F(x)$ (cdf) has a simple formulation:

For $x < 0$:

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0$$

For $x \ge 0$:

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt = \int_0^x \frac{1}{\alpha} e^{-t/\alpha} dt = \left[-e^{-t/\alpha}\right]_0^x = -e^{-x/\alpha} + 1 = 1 - e^{-x/\alpha}$$

$$\Rightarrow F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x/\alpha} & x \ge 0 \end{cases}$$

Example 5.26 (Library arrivals, cont'd)**.** Recall the example the arrival rate of students at Parks library between 12:00 and 12:10pm early in the week to be about 12.5 students per minute. That translates to a $1/12.5 = .08$ minute average waiting time between student arrivals.

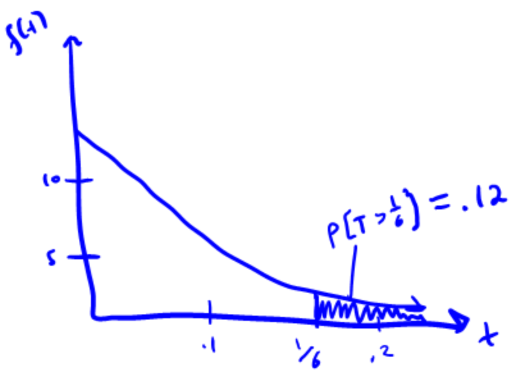
Consider observing the entrance to Parks library at exactly noon next Tuesday and define the random variable

$$T = \text{the waiting time (min) until the first student passes through the door.}$$

$\alpha = .08$

Using $T \sim \text{Exp}(.08)$, what is the probability of waiting more than 10 seconds (1/6 min) for the first arrival?

$P(\text{waiting more than 10 seconds}) = P(T > \frac{1}{6})$
$= 1 - P[T \le \frac{1}{6}]$
$= 1 - (1 - e^{-\frac{1}{6(.08)}}) = .12$



What is the probability of waiting less than 5 seconds? (5 seconds $= \frac{1}{12}$ minute):

$P(T < \frac{1}{12}) = P(T \le \frac{1}{12}) = F(\frac{1}{12}) = (1 - e^{-\frac{1}{12 \cdot .08}}) \approx .6471$

Recall: Since T is continuous, $P(T = c) = 0$ for all c, meaning $P(T < c) = P(T \le c)$

5.2.5 The Normal distribution

Normal qq plot

We have already seen the normal distribution as a "bell shaped" distribution, but we can formalize this.

Definition 5.19. The *normal* or *Gaussian*(μ, σ^2) distribution is a continuous probability distribution with probability density

2 parameters

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} \qquad \text{for all } x \in \mathbb{R}$$

for $\sigma > 0$. and $\mu \in \mathbb{R}$ (literally any number)

A normal random variable is (often) a finite average of many repeated, independent, identical trials.

- Mean width of the next 50 hexamine pellets
- Mean height of 30 students
- Total % yield of the next 40 runs of a chemical process.

It is not obvious, but

1. $\int\limits_{-\infty}^{\infty} f(x)dx = \int\limits_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx =$

2. $\mathrm{E}X = \int\limits_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx =$

3. $\mathrm{Var}X = \int\limits_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx =$

The Calculus I methods of evaluating integrals via anti-differentiation will fail when it comes to normal densities. They do not have anti-derivatives that are expressible in terms of elementary functions.

The use of tables for evaluating normal probabilities depends on the following relationship. If $X \sim \text{Normal}(\mu, \sigma^2)$,

$$P[a \le X \le b] = \int_a^b \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx = \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = P\left[\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right]$$

where $Z \sim \text{Normal}(0, 1)$.

Definition 5.20. The normal distribution with $\mu = 0$ and $\sigma = 1$ is called the *standard normal distribution.*

So, we can find probabilities for all normal distributions by tabulating probabilities for only the standard normal distribution. We will use a table of the **standard normal cumulative probability function**.

$$\Phi(z) = F(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2} dt.$$

Example 5.27 (Standard normal probabilities). $P[Z < 1.76]$

$P[.57 < Z < 1.32]$

We can also do it in reverse, find z such that $P[-z < Z < z] = .95$.

Example 5.28 (Baby food)**.** J. Fisher, in his article Computer Assisted Net Weight Control (*Quality Progress*, June 1983), discusses the filling of food containers with strained plums and tapioca by weight. The mean of the values portrayed is about 137.2g, the standard deviation is about 1.6g, and data look bell-shaped. Let

$$W = \text{the next fill weight.}$$

Let's find the probability that the next jar contains less food by mass than it's supposed to (declared weight = 135.05g).

Table B.3
Standard Normal Cumulative Probabilities

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt$$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985

Example 5.29 (More normal probabilities)**.** Using the standard normal table, calculate the following:

$P(X \leq 3), X \sim \text{Normal}(2, 64)$

$P(X > 7), X \sim \text{Normal}(6, 9)$

$P(|X - 1| > 0.5), X \sim \text{Normal}(2, 4)$

We can find standard normal quantiles by using the standard normal table in reverse.

Example 5.30 (Baby food, cont'd)**.** For the jar weights $X \sim \text{Normal}(137.2, 1.62^2)$, find $Q(0.1)$.

Table B.3

Standard Normal Cumulative Probabilities

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt$$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985

Example 5.31 (Normal quantiles)**.** Find:

$Q(0.95)$ of $X \sim \text{Normal}(9, 3)$.

c such that $P(|X - 2| > c) = 0.01$, $X \sim \text{Normal}(2, 4)$

5.3 Joint distributions and independence (discrete)

Most applications of probability to engineering statistics involve not one but several random variables. In some cases, the application is intrinsically multivariate.

Example 5.32. Consider the assembly of a ring bearing with nominal inside diameter 1.00 in. on a rod with nominal diameter .99 in. If

$$X = \text{the ring bearing inside diameter}$$
$$Y = \text{the rod diameter}$$

One might be interested in

$$P[\text{there is an interference in assembly}] =$$

Even when a situation is univariate, samples larger than size 1 are essentially always used in engineering applications. The n data values in a sample are usually thought of as subject to chance and their simultaneous behavior must then be modeled.

This is actually a very broad and difficult subject, we will only cover a brief introduction to the topic: **jointly discrete random variables**.

5.3.1 Joint distributions

For several discrete random variable, the device typically used to specify probabilities is a *joint probability function.* The two-variable version of this is defined.

Definition 5.21. A *joint probability function (joint pmf)* for discrete random variables X and Y is a nonnegative function $f(x, y)$, giving the probability that (simultaneously) X takes the values x and Y takes the values y. That is,

$$f(x, y) = P[X = x \text{ and } Y = y]$$

Properties:

1.

2.

For the discrete case, it is useful to give $f(x, y)$ in a **table**.

Example 5.33 (Two bolt torques, cont'd)**.** Recall the example of measure the bolt torques on the face plates of a heavy equipment component to the nearest integer. With

$$X = \text{the next torque recorded for bolt 3}$$
$$Y = \text{the next torque recorded for bolt 4}$$

the joint probability function, $f(x, y)$, is

y\x	11	12	13	14	15	16	17	18	19	20
20	0	0	0	0	0	0	0	2/34	2/34	1/34
19	0	0	0	0	0	0	2/34	0	0	0
18	0	0	1/34	1/34	0	0	1/34	1/34	1/34	0
17	0	0	0	0	2/34	1/34	1/34	2/34	0	0
16	0	0	0	1/34	2/34	2/34	0	0	2/34	0
15	1/34	1/34	0	0	3/34	0	0	0	0	0
14	0	0	0	0	1/34	0	0	2/34	0	0
13	0	0	0	0	1/34	0	0	0	0	0

$P[X = 18 \text{ and } Y = 17]$

$P[X = 14 \text{ and } Y = 19]$

By summing up certain values of $f(x, y)$, probabilities associated with X and Y with patterns of interest can be obtained.

Consider:

$$P[X \geq Y]$$

y\x	11	12	13	14	15	16	17	18	19	20
20										
19										
18										
17										
16										
15										
14										
13										

$$P[|X - Y| \leq 1]$$

y\x	11	12	13	14	15	16	17	18	19	20
20										
19										
18										
17										
16										
15										
14										
13										

$$P[X = 17]$$

y\x	11	12	13	14	15	16	17	18	19	20
20										
19										
18										
17										
16										
15										
14										
13										

5.3.2 Marginal distributions

In a bivariate problem, one can add down columns in the (two-way) table of $f(x,y)$ to get values for the probability function of X, $f_X(x)$ and across rows in the same table to get values for the probability distribution of Y, $f_Y(y)$.

Definition 5.22. The individual probability functions for discrete random variables X and Y with joint probability function $f(x,y)$ are called *marginal probability functions.* They are obtained by summing $f(x,y)$ values over all possible values of the other variable.

$$f_X(x) = \sum_y f(x,y)$$
$$f_Y(y) = \sum_x f(x,y)$$

Example 5.34 (Torques, cont'd)**.** Find the marginal probability functions for X and Y from the following joint pmf.

y\x	11	12	13	14	15	16	17	18	19	20
20	0	0	0	0	0	0	0	2/34	2/34	1/34
19	0	0	0	0	0	0	2/34	0	0	0
18	0	0	1/34	1/34	0	0	1/34	1/34	1/34	0
17	0	0	0	0	2/34	1/34	1/34	2/34	0	0
16	0	0	0	1/34	2/34	2/34	0	0	2/34	0
15	1/34	1/34	0	0	3/34	0	0	0	0	0
14	0	0	0	0	1/34	0	0	2/34	0	0
13	0	0	0	0	1/34	0	0	0	0	0

Getting marginal probability functions from joint probability functions begs the question whether the process can be reversed. **Can we find joint probability functions from marginal probability functions?**

5.3.3 Conditional distributions

When working with several random variables, it is often useful to think about what is expected of one of the variables, given the values assumed by all others.

Definition 5.23. For discrete random variables X and Y with joint probability function $f(x, y)$, the *conditional probability function of* X *given* $Y = y$ is the function of x

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f(x,y)}{\sum\limits_x f(x,y)}$$

and the *conditional probability function of* Y *given* $X = x$ is the function of y

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{f(x,y)}{\sum\limits_y f(x,y)}.$$

Example 5.35 (Torque, cont'd)**.** For the torque example with the following joint distribution, find the following:

1. $f_{Y|X}(20|18)$
2. $f_{Y|X}(y|15)$
3. $f_{Y|X}(y|20)$
4. $f_{X|Y}(x|18)$

$y \backslash x$	11	12	13	14	15	16	17	18	19	20	$f_Y(y)$
20	0/34	0/34	0/34	0/34	0/34	0/34	0/34	2/34	2/34	1/34	5/34
19	0/34	0/34	0/34	0/34	0/34	0/34	2/34	0/34	0/34	0/34	2/34
18	0/34	0/34	1/34	1/34	0/34	0/34	1/34	1/34	1/34	0/34	5/34
17	0/34	0/34	0/34	0/34	2/34	1/34	1/34	2/34	0/34	0/34	6/34
16	0/34	0/34	0/34	1/34	2/34	2/34	0/34	0/34	2/34	0/34	7/34
15	1/34	1/34	0/34	0/34	3/34	0/34	0/34	0/34	0/34	0/34	5/34
14	0/34	0/34	0/34	0/34	1/34	0/34	0/34	2/34	0/34	0/34	3/34
13	0/34	0/34	0/34	0/34	1/34	0/34	0/34	0/34	0/34	0/34	1/34
$f_X(x)$	1/34	1/34	1/34	2/34	9/34	3/34	4/34	7/34	5/34	1/34	34/34

5.3.4 Independence

Recall the following joint distribution:

$y\backslash x$	1	2	3	$f_Y(y)$
3	0.08	0.08	0.04	0.20
2	0.16	0.16	0.08	0.40
1	0.16	0.16	0.08	0.40
$f_X(x)$	0.40	0.40	0.20	1.00

What do you notice?

Definition 5.24. Discrete random variables X and Y are *independent* if their joint distribution function $f(x, y)$ is the product of their respective marginal probability functions. This is, independence means that

$$f(x, y) = f_X(x)f_Y(y) \qquad \text{for all } x, y.$$

If this does not hold, then X and Y are *dependent.*

Alternatively, discrete random variables X and Y are independent if for all x and y,

If X and$Y$ are not only independent but also have the same marginal distribution, then they are **independent and identically distributed (iid)**.

5.4 Functions of several random variables

We've now talked about ways to simultaneously model several random variables. An important engineering use of that material is in the analysis of system output that are functions of random inputs.

5.4.1 Linear combinations

For engineering purposes, it often suffices to know the mean and variance for a function of several random variables, $U = g(X_1, X_2, \ldots, X_n)$ (as opposed to knowing the whole distribution of U). When g is **linear**, there are explicit functions.

Proposition 5.1. *If $X_1, X_2, \ldots, X_n$ are n independent random variables and $a_0, a_1, \ldots, a_n$ are $n+1$ constants, then the random variable $U = a_0 + a_1X_1 + a_2X_2 + \cdots + a_nX_n$ has mean*

$$EU = a_0 + a_1EX_1 + a_2EX_2 + \cdots + a_nEX_3$$

and variance

$$VarU = a_1^2\,VarX_1 + a_2^2\,VarX_2 + \cdots + a_n^2\,VarX_3$$

Example 5.36. Say we have two independent random variables X and Y with $\mathrm{E}X = 3.3, \mathrm{Var}X = 1.91, \mathrm{E}Y = 25$, and $\mathrm{Var}Y = 65$. Find the mean and variance for

$$
\begin{aligned}
U &= 3 + 2X - 3Y \\
V &= -4X + 3Y \\
W &= 2X - 5Y \\
Z &= -4X - 6Y
\end{aligned}
$$

Example 5.37. Say $X \sim Binomial(n = 10, p = 0.5)$ and $Y \sim Poisson(\lambda = 3)$. Calculate the mean and variance of $Z = 5 + 2X - 7Y$.

A particularly important use of Proposition 5.1 concerns n iid random variables where each $a_i = \frac{1}{n}$.

We can find the mean and variance of the random variable

$$\overline{X} = \frac{1}{n}X_1 + \cdots \frac{1}{n}X_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

as they relate to the population parameters $\mu = \mathrm{E}X_i$ and $\sigma^2 = \mathrm{Var}X_i$.

For independent variables $X_1, \ldots, X_n$ with common mean μ and variance σ^2,

$\mathrm{E}\overline{X} =$

$\mathrm{Var}\overline{X} =$

Example 5.38 (Seed lengths)**.** One botanist measured the length of 10 seeds from the same plant. The seed lengths measurements are $X_1, X_2, \ldots, X_{10}$. Suppose it is known that the seed lengths are iid with mean $\mu = 5$ mm and variance $\sigma^2 = 2$mm.

Calculate the mean and variance of the average of 10 seed measurements.

5.4.2 Central limit theorem

One of the most frequently used statistics in engineering applications is the sample mean. We can relate the mean and variance of the probability distribution of the sample mean to those of a single observation when an iid model is appropriate.

Proposition 5.2. *If $X_1, \dots, X_n$ are iid random variable (with mean μ and variance σ^2), then for large n, the variable $\overline{X}$ is approximately normally distributed. That is,*

$$\overline{X} \dot\sim Normal\left(\mu, \frac{\sigma^2}{n}\right)$$

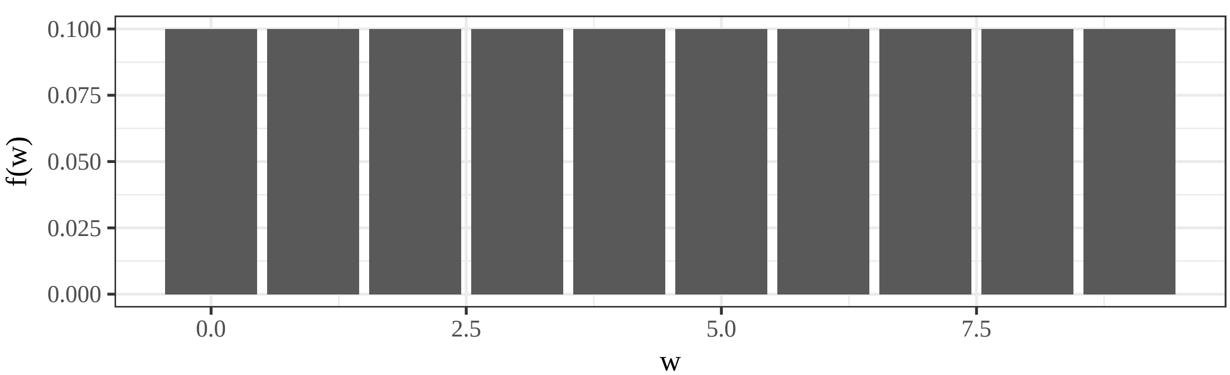
This is one of the **most important** results in statistics.

Example 5.39 (Tool serial numbers)**.** Consider selecting the last digit of randomly selected serial numbers of pneumatic tools. Let

$$\begin{aligned} W_1 = & \text{ the last digit of the serial number observed next Monday at 9am} \\ W_2 = & \text{ the last digit of the serial number observed the following Monday at 9am} \end{aligned}$$

A plausible model for the pair of random variables W_1, W_2 is that they are independent, each with the marginal probability function

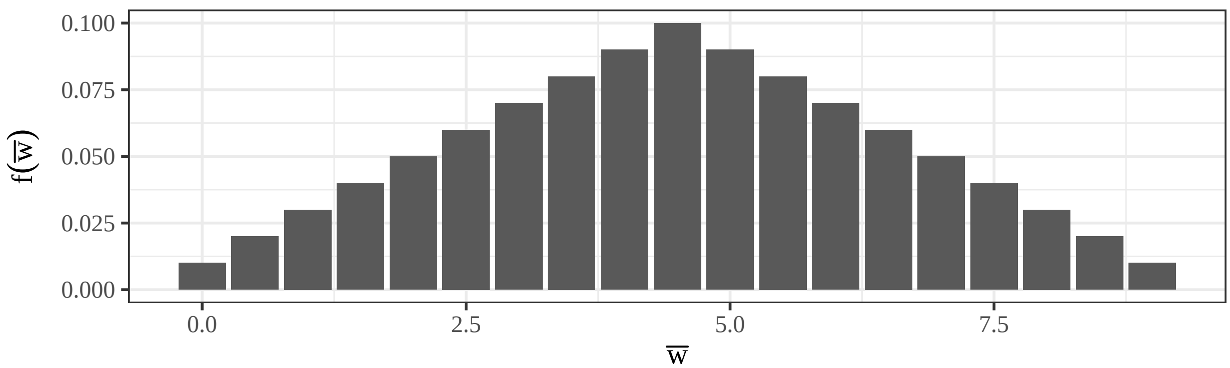
$$f(w) = \begin{cases} .1 & w = 0, 1, 2, \dots, 9 \\ 0 & \text{otherwise} \end{cases}$$



With E$W = 4.5$ and Var$W = 8.25$.

Using such a distribution, it is possible to see that $\overline{W} = \frac{1}{2}(W_1 + W_2)$ has probability distribution

$\overline{w}$	$f(\overline{w})$	$\overline{w}$	$f(\overline{w})$	$\overline{w}$	$f(\overline{w})$	$\overline{w}$	$f(\overline{w})$	$\overline{w}$	$f(\overline{w})$
0.00	0.01	2.00	0.05	4.00	0.09	6.00	0.07	8	0.03
0.50	0.02	2.50	0.06	4.50	0.10	6.50	0.06	8.5	0.02
1.00	0.03	3.00	0.07	5.00	0.09	7.00	0.05	9	0.01
1.50	0.04	3.50	0.08	5.50	0.08	7.50	0.04		



Comparing the two distributions, it is clear that even for a completely flat/uniform distribution of W and a small sample size of $n = 2$, the probability distribution of W looks more bell-shaped than the underlying distribution.

Now consider larger and larger sample sizes, $n = 1, \ldots, 40$:

Click for video...

Example 5.40 (Stamp sale time)**.** Imagine you are a stamp salesperson (on eBay). Consider the time required to complete a stamp sale as S, and let

$$\overline{S} = \text{the sample mean time required to complete the next 100 sales}$$

Each individual sale time should have an $Exp(\alpha = 16.5s)$ distribution. We want to consider approximating $P[\overline{S} > 17]$.

Example 5.41 (Cars)**.** Suppose a bunch of cars pass through certain stretch of road. Whenever a car comes, you look at your watch and record the time. Let X_i be the time (in minutes) between when the i^{th} car comes and the $(i+1)^{th}$ car comes for $i = 1, \ldots, 44$. Suppose you know the average time between cars is 1 minute. Find the probability that the average time gap between cars exceeds 1.05 minutes.

Example 5.42 (Baby food jars, cont'd)**.** The process of filling food containers appears to have an inherent standard deviation of measured fill weights on the order of $1.6g$. Suppose we want to calibrate the filling machine by setting an adjustment knob and filling a run of n jars. Their sample mean net contents will serve as an indication of the process mean fill level corresponding to that knob setting.

You want to choose a sample size, n, large enough that there is an 80% chance the sample mean is within .3g of the actual process mean.

Example 5.43 (Printing mistakes)**.** Suppose the number of printing mistakes on a page follows some unknown distribution with a mean of 4 and a variance of 9. Assume that number of printing mistakes on a printed page are iid.

1. What is the approximate probability distribution of the average number of printing mistakes on 50 pages?

2. Can you find the probability that the number of printing mistakes on a single page is less than 3.8?

3. Can you find the probability that the average number of printing mistakes on 10 pages is less than 3.8?

4. Can you find the probability that the average number of printing mistakes on 50 pages is less than 3.8?

Table B.3
Standard Normal Cumulative Probabilities

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt$$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2297	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Table B.3

Standard Normal Cumulative Probabilities (*continued*)

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9773	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9983	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

This table was generated using MINITAB.